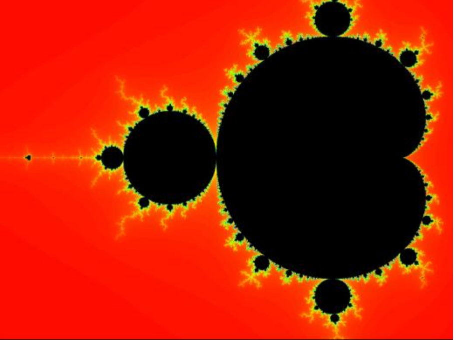

Generalised hydrodynamics of the KdV soliton gas

Dynamics Days Europe
MS: Recent advances in dispersive and generalised hydrodynamics

Thibault Bonnemain, 8th September 2023

[joint work with Benjamin Doyon and Gennady El]

Gibbs ensembles and Hydrodynamics

- Boltzmann 1868: micro-canonical ensemble in long time limit

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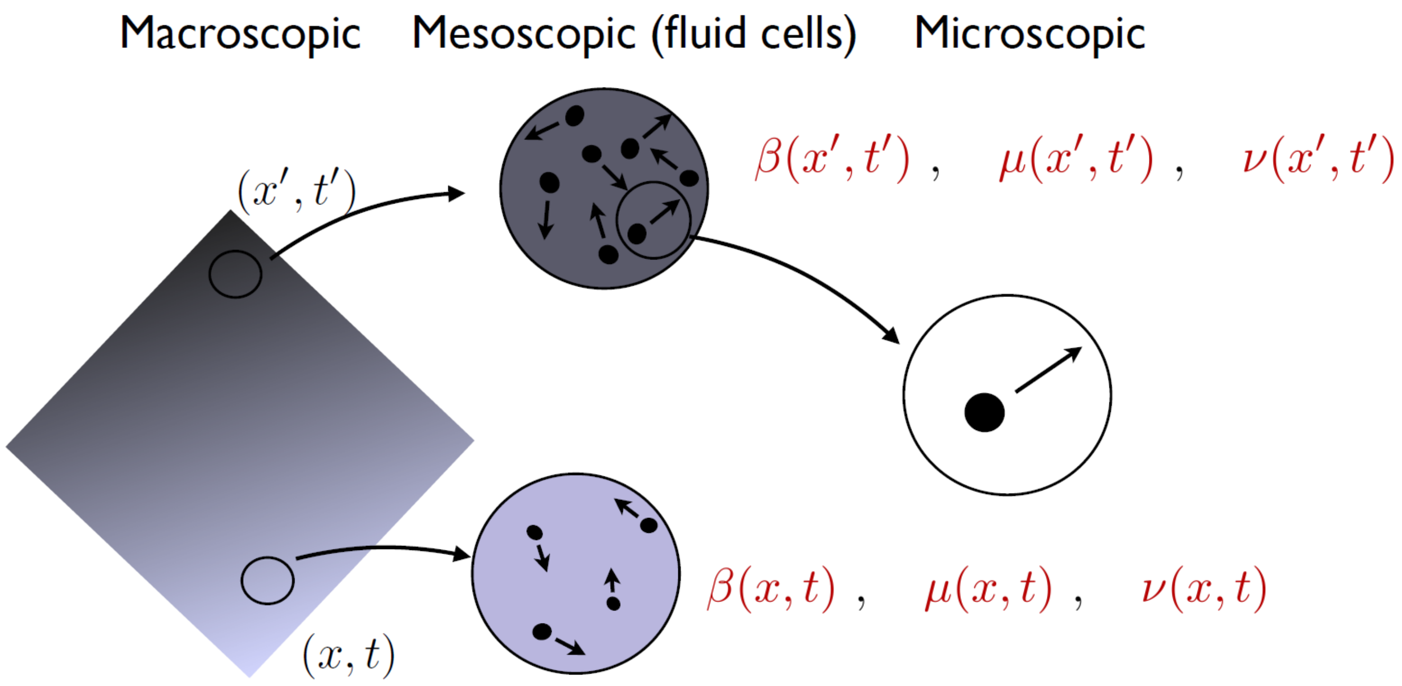

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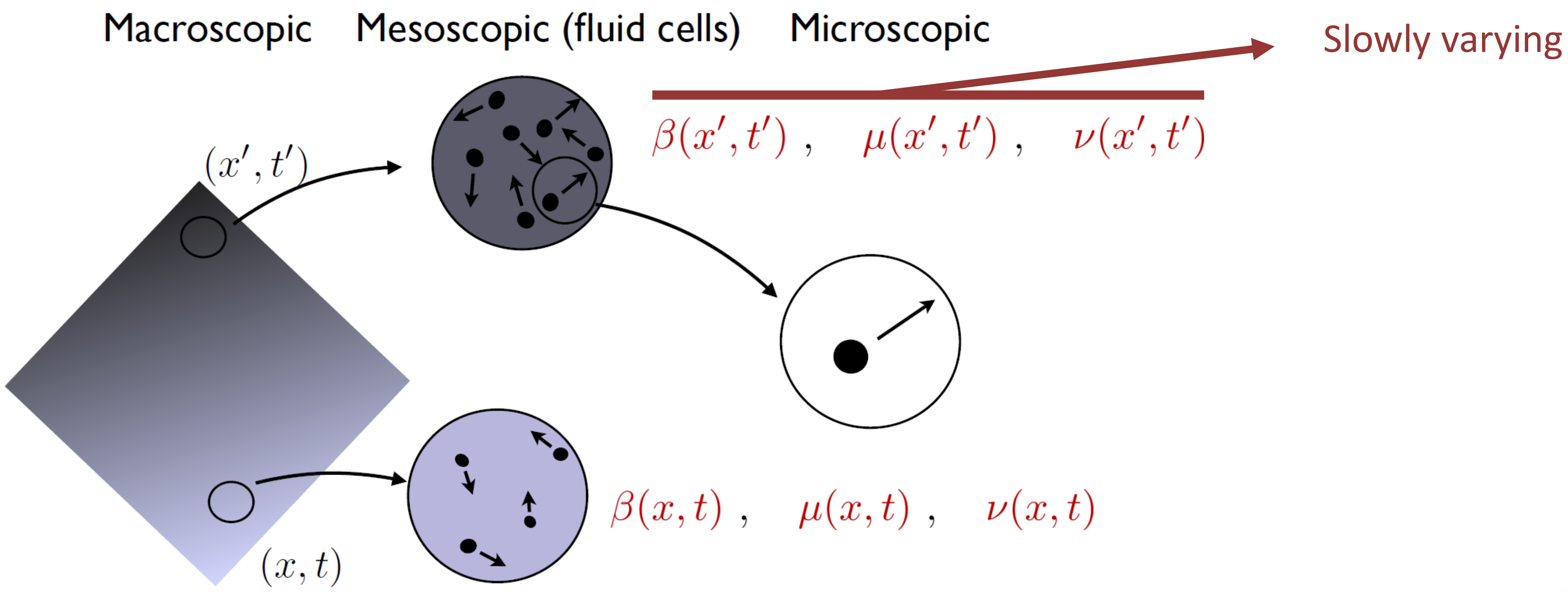

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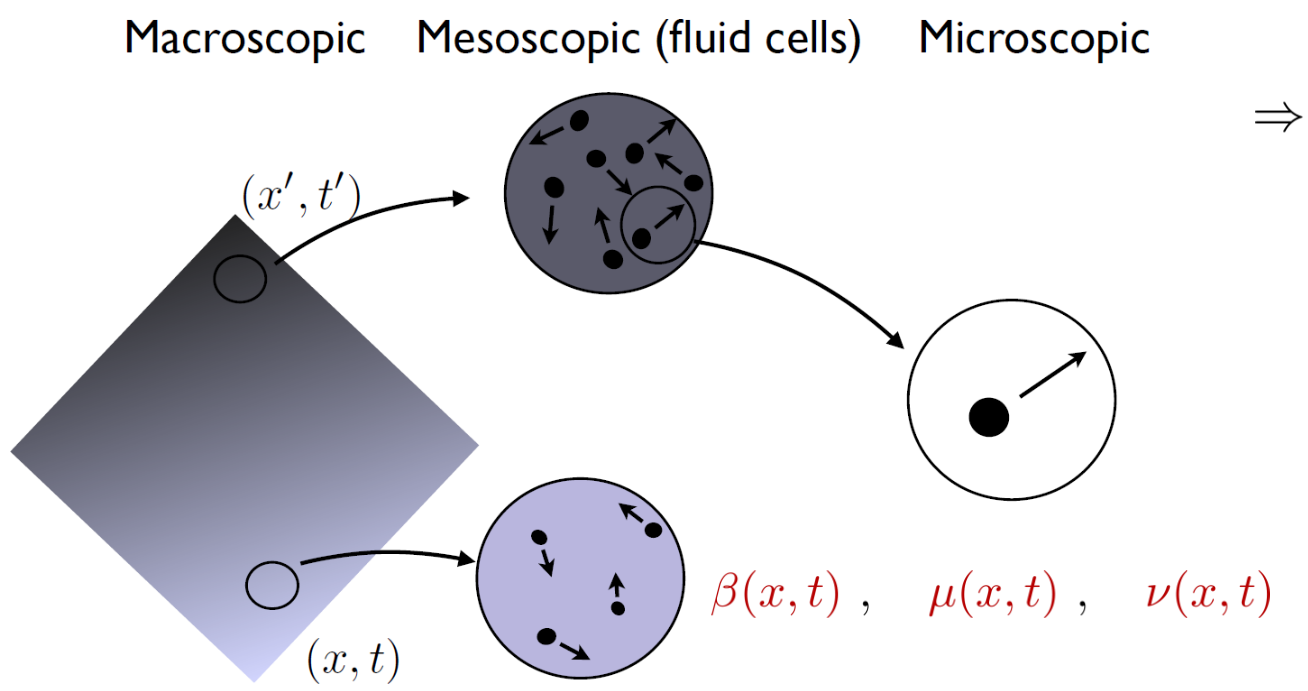

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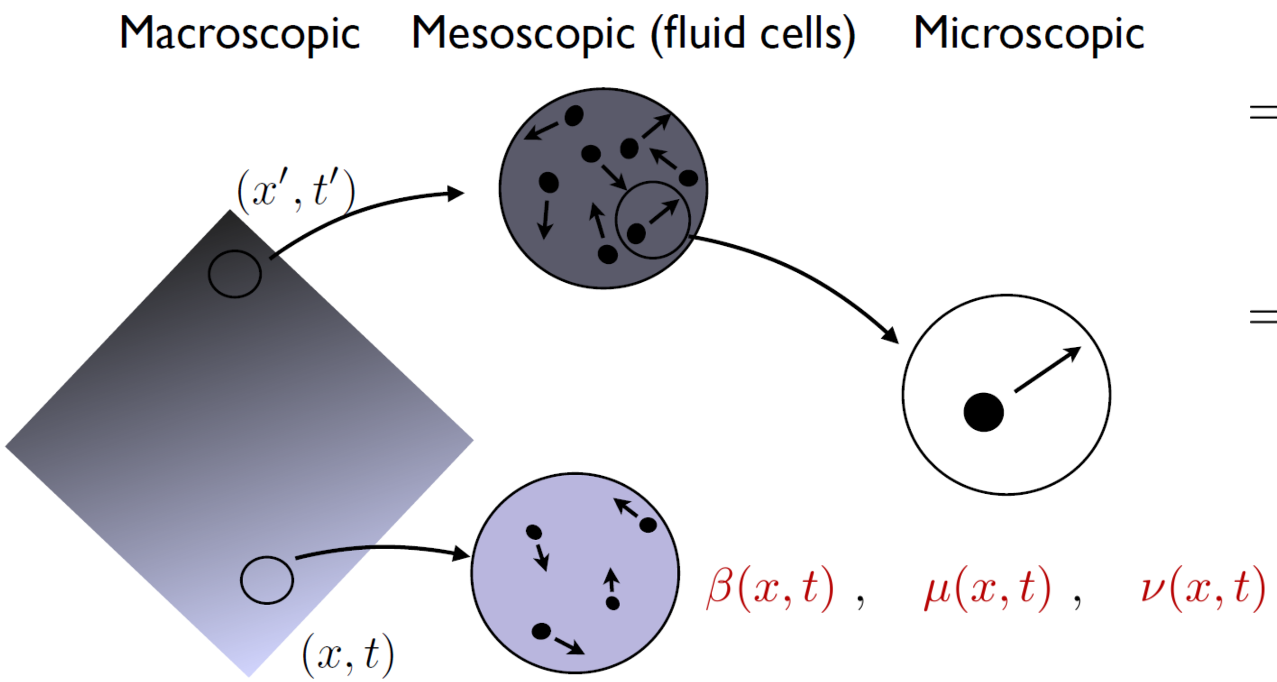

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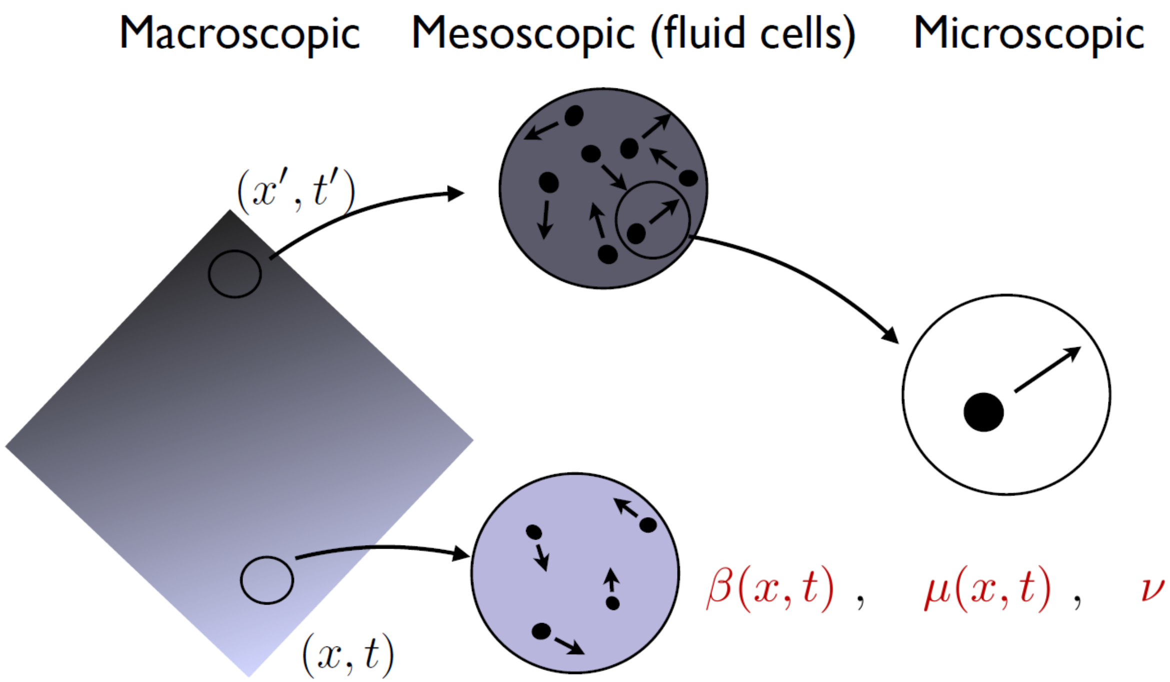

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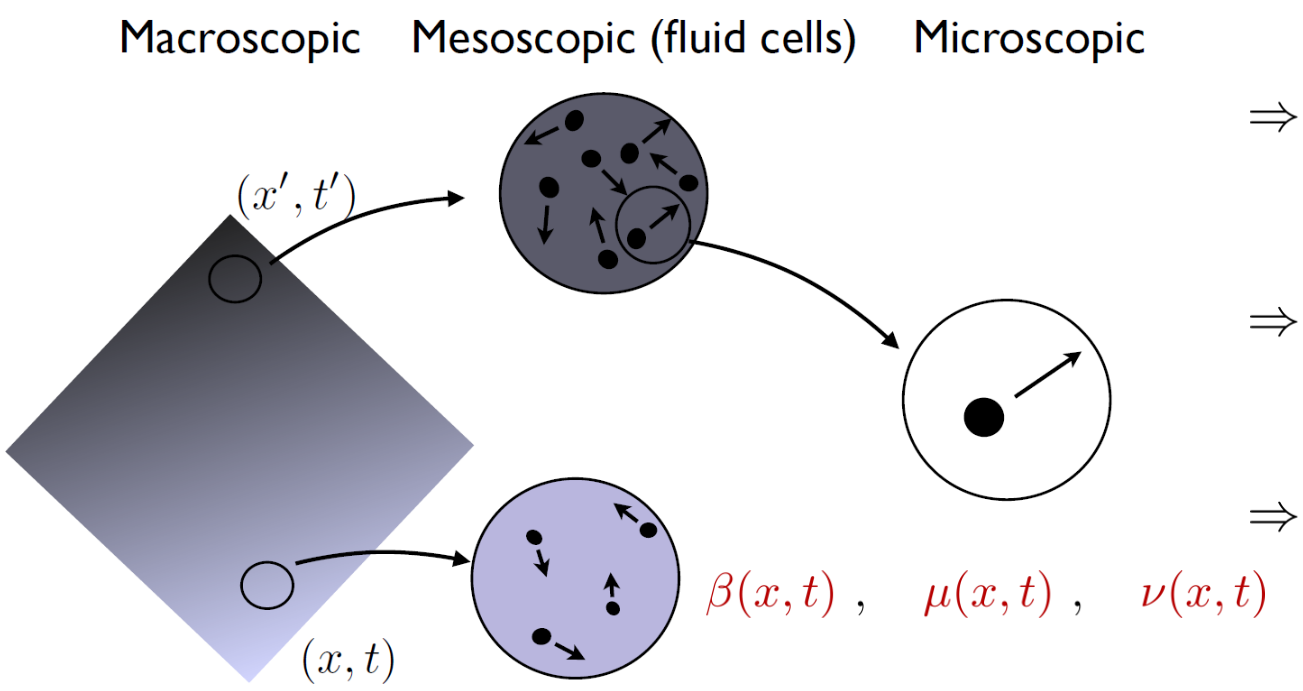

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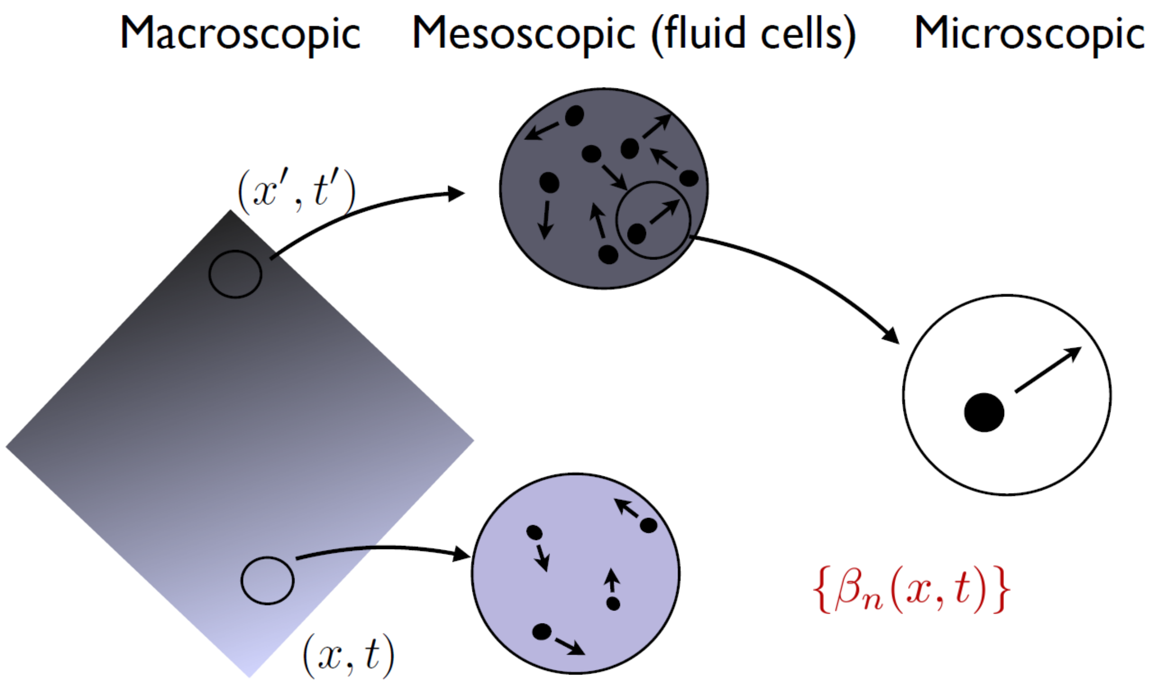

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- KdV: integrable nonlinear dispersive PDE

$$\partial_t \varphi + 6\varphi\partial_x\varphi + \partial_x^3\varphi = 0 \ .$$

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- Infinite set of conservation laws

Time conserved "charges"

$$Q_n = \int \mathrm{d}x \ q_n(x,t) \ , \quad J_n = \int \mathrm{d}t \ j_n(x,t) \ ,$$

Space conserved "currents"

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[Miura, Gardner, Kruskal (1968)]

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- Solvable via Inverse Scattering Transfom.

[Gardner, Greene, Kruskal, Miura (1967)]

GHD from scattering theory

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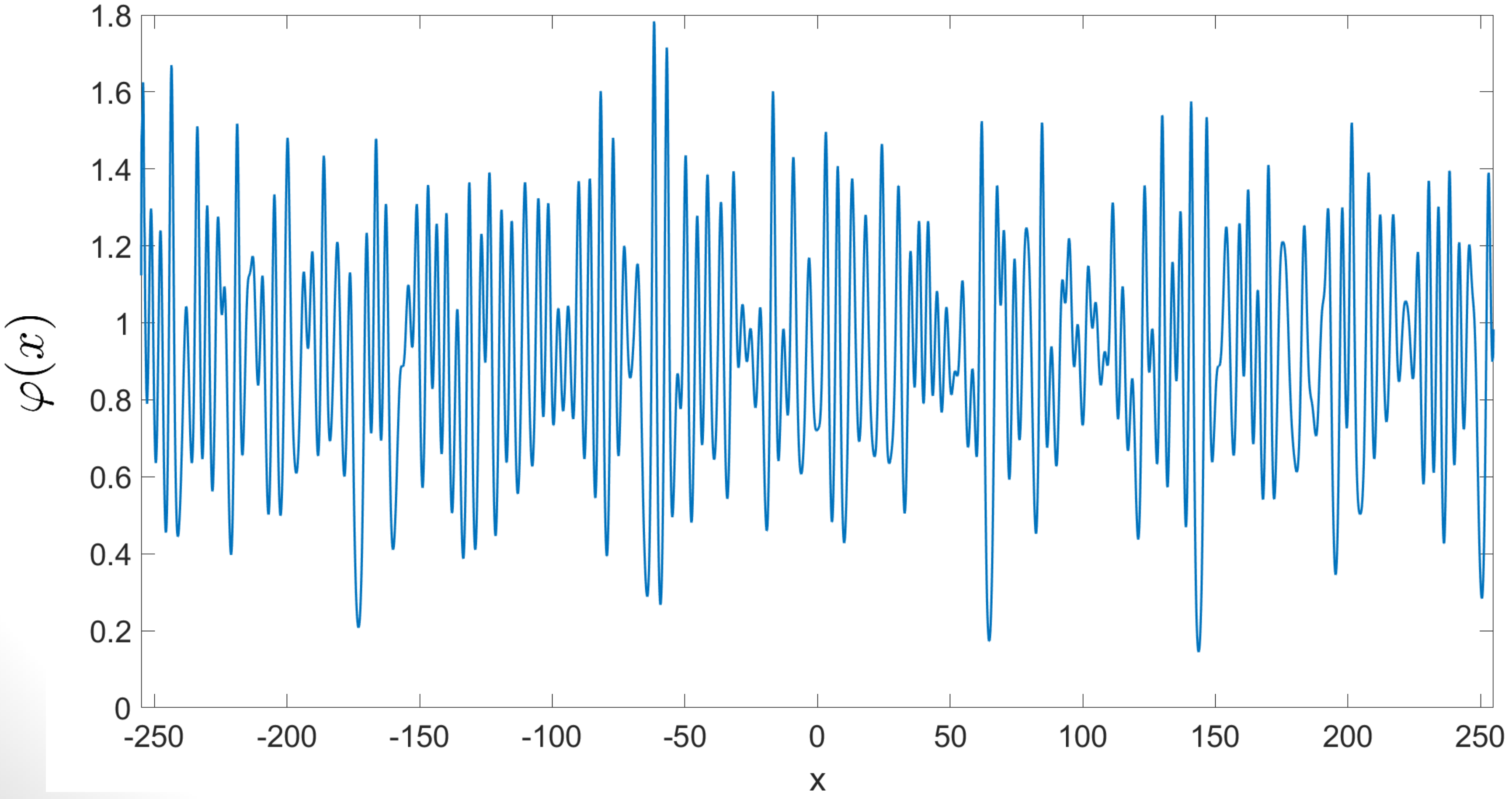

Single realisation of a KdV soliton gas

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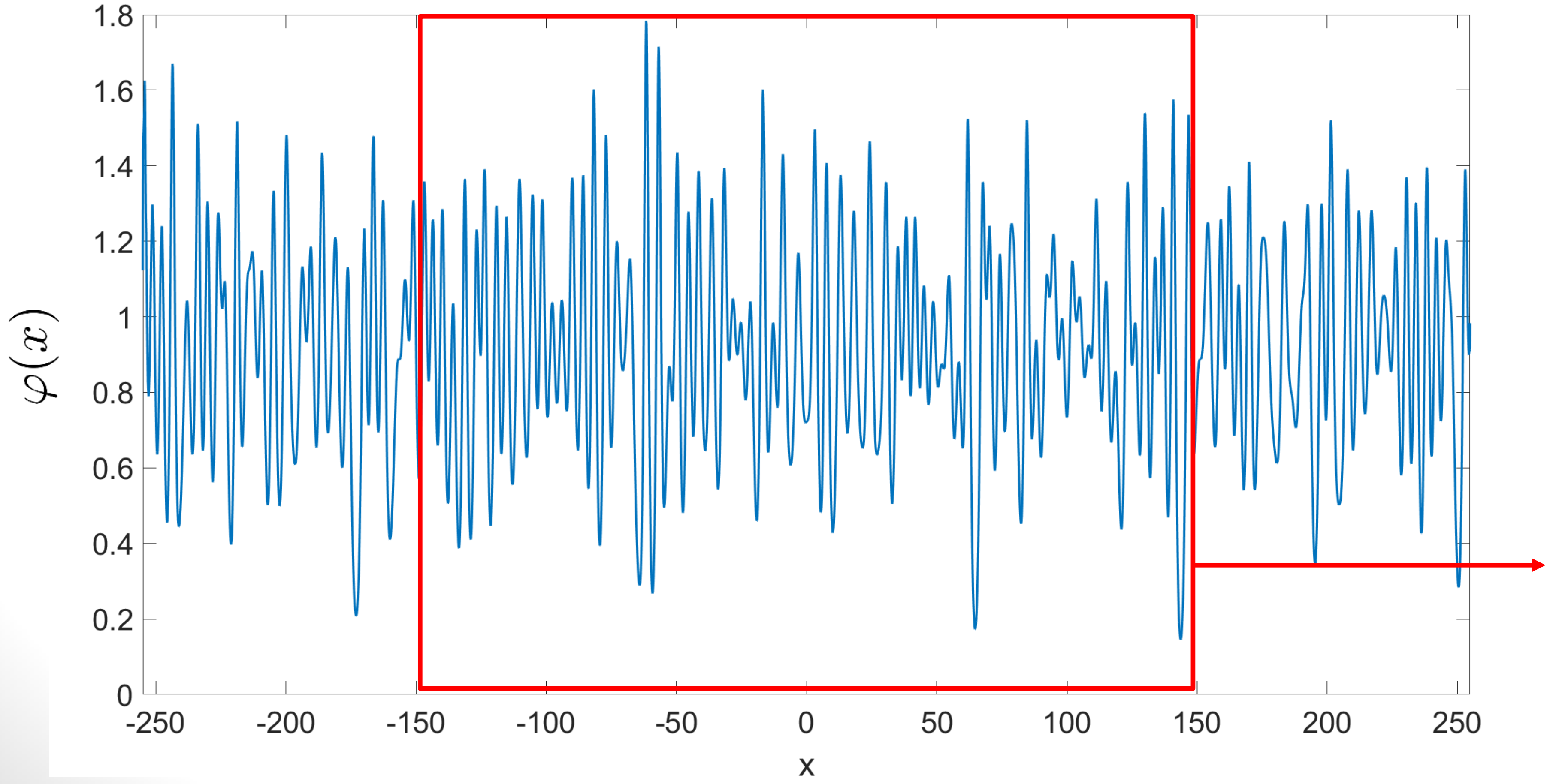

Single realisation of a KdV soliton gas

Fluid cell of size L characterised by local GGE

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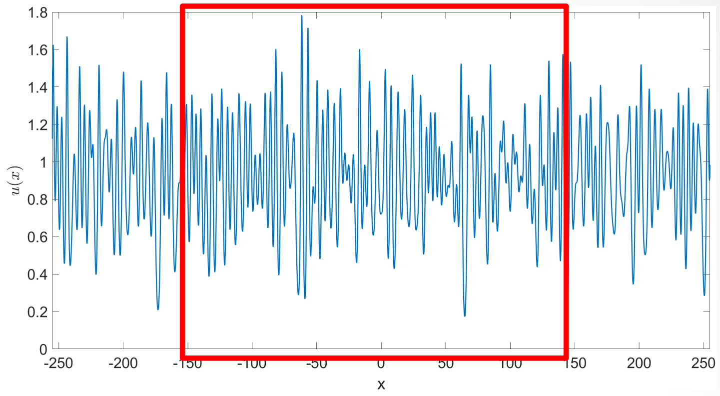
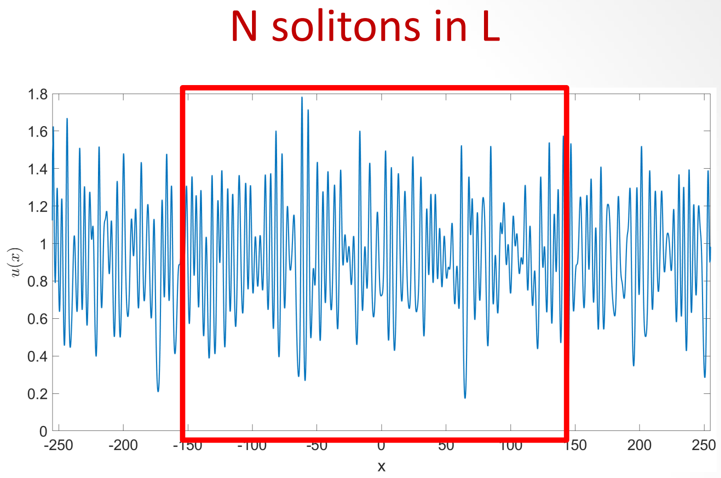
$$\varphi_N \sim \sum_{i=1}^{N} 2\eta_i^2 \mathrm{sech}^2 \left[\eta_i \left(x - 4\eta_i^2 t - x_i^{\pm} \right) \right] \quad \text{as} \quad t \to \pm\infty.$$

N solitons in L



[Zakharov (1971)]

GHD from scattering theory



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Action coordinate

Angle coordinate

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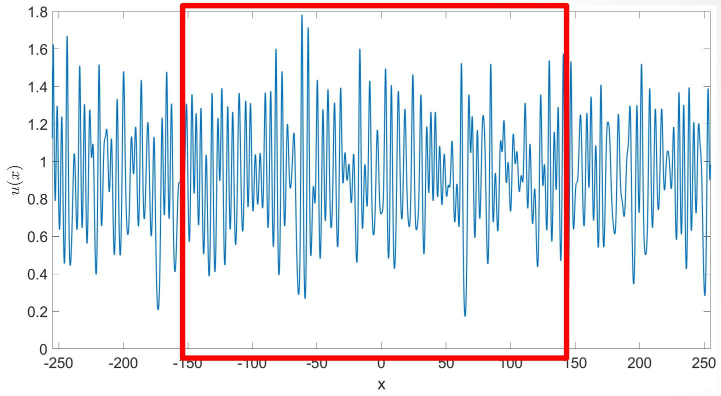

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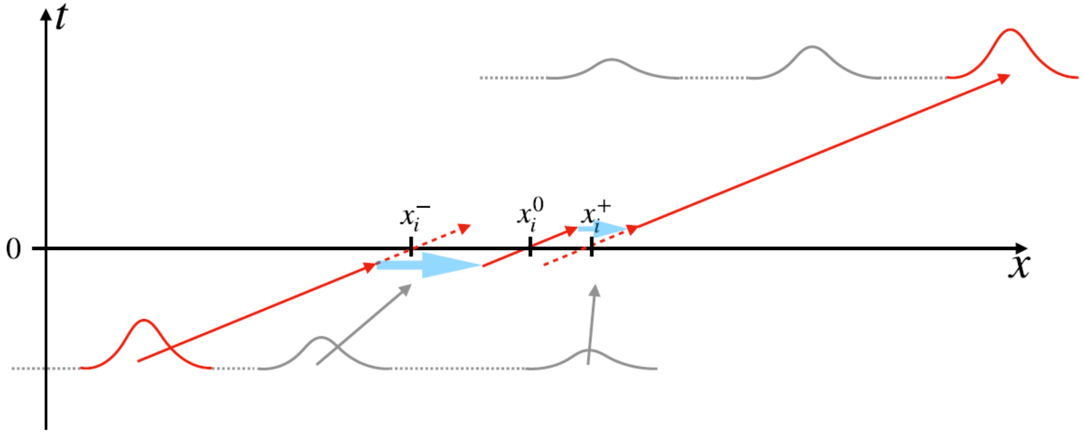

Scattering is elastic and
2-body factorisable

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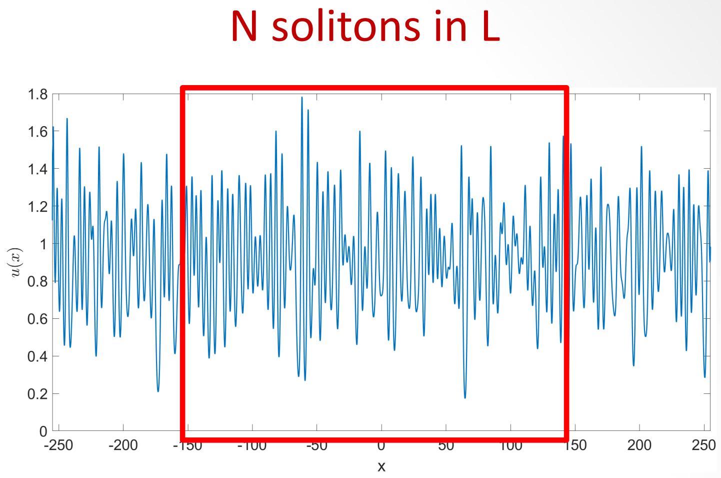

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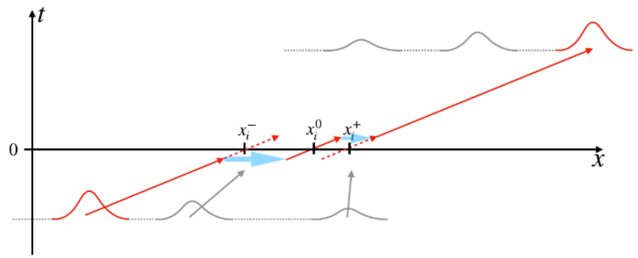
- Relation between asymptotic states given by scattering shift



$$x_i^+ - x_i^- = \sum_j \frac{\mathrm{sgn}(\eta_\mathrm{i} - \eta_\mathrm{j})}{\eta_i} \ln \left|\frac{\eta_i + \eta_j}{\eta_i - \eta_j}\right|$$

[Lax (1968)]

Thermodynamics

- Partition function

$$\mathcal{Z} = \sum_{N=0}^{\infty} \frac{1}{N!} \int_{\Gamma^N \times \mathbb{R}^N} \prod_{i=1}^{N} \frac{\mathrm{d}p(\eta_i)}{2\pi} \mathrm{d}x_i^- \exp\left[-\sum_{i=1}^{N} w(\eta_i)\right] \chi\left(\varphi_N(x, t=0) < \epsilon_x, x \notin [0, L]\right)$$

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$$p(\eta) = 4\eta^2$$

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Constraint / Entropy

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- NDR of soliton gases

$$\sigma(\eta)\rho(\eta) = \eta - \int_{\Gamma} \mathrm{d}\mu \; \rho(\mu) \log\left|\frac{\eta+\mu}{\eta-\mu}\right|$$

$$\rho(\eta)\mathrm{d}\eta\mathrm{d}x = \# \text{ of solitons in } [x, x+\mathrm{d}x] \times [\eta, \eta+\mathrm{d}\eta]$$

Density of States

[El (2003)]

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Density of States

[El (2003)]

- Alternative interpretation

$$\frac{\mathrm{d}x^-(\eta)}{\mathrm{d}x} = \frac{\sigma(\eta)\rho(\eta)}{\eta}$$

Change of metric

Some more (maybe) familiar relations

- Spectral scaling function

$$\log\left[\frac{4\sigma(\eta)}{\pi}\right] = w(\eta) - \int_\Gamma \mathrm{d}\mu \frac{1}{\sigma(\mu)} \log\left|\frac{\eta-\mu}{\eta+\mu}\right|, \qquad \mathcal{F} = -\int_\Gamma \frac{\eta \mathrm{d}\eta}{\sigma(\eta)}$$

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$$\mathsf{C}_{ab} \equiv \int_\Gamma \mathrm{d}x\ \left(\langle q_a(x) q_b(0)\rangle - \langle q_a(x)\rangle\langle q_b(0)\rangle\right) = -\frac{\partial^2 \mathcal{F}}{\partial \beta_a \partial \beta_b}$$

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[Doyon (2018)]

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[Doyon (2018)]

$$h^{\mathrm{dr}}(\eta) = h(\eta) + \int_\Gamma \frac{\mathrm{d}\mu}{\sigma(\mu)} \log\left|\frac{\eta-\mu}{\eta+\mu}\right| h^{\mathrm{dr}}(\mu)$$

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Statistical factor

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Some more (maybe) familiar relations

- Spectral scaling function

$$\log\left[\frac{4\sigma(\eta)}{\pi}\right] = w(\eta) - \int_\Gamma \mathrm{d}\mu \frac{1}{\sigma(\mu)} \log\left|\frac{\eta-\mu}{\eta+\mu}\right|, \qquad \mathcal{F} = -\int_\Gamma \frac{\eta \mathrm{d}\eta}{\sigma(\eta)}$$

- Density of states

$$\rho(\eta) = \frac{\delta \mathcal{F}}{\delta w(\eta)}$$

- Thermodynamic averages

$$\langle q_n \rangle = \int_\Gamma \mathrm{d}\eta\ \rho(\eta) h_n(\eta) = \frac{\partial \mathcal{F}}{\partial \beta_n}$$

- Correlations

	MB	FD	BE	Simulations
C_{00}^{DC}	0.0235	-2.28	2.32	0.022 ± 0.003
C_{01}^{DC}	0.027	-3.18	3.23	0.024 ± 0.004
C_{11}^{DC}	0.042	-4.48	4.56	0.039 ± 0.005
C_{00}^{U}	0.22	0.028	0.41	0.2 ± 0.03
C_{01}^{U}	0.28	0.072	0.49	0.23 ± 0.04
C_{11}^{U}	0.39	0.12	0.66	0.36 ± 0.05

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- Integrability: infinite number of conservation laws

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$$v^{\mathrm{eff}}(\eta) = 4\eta^2 + \frac{1}{\eta}\int_\Gamma \log\left|\frac{\eta+\mu}{\eta-\mu}\right| \rho(\mu)[v^{\mathrm{eff}}(\eta) - v^{\mathrm{eff}}(\mu)]\mathrm{d}\mu$$

Recap

Gibbs weights e.g. $\quad w(\eta) = \sum_k \beta_k \eta^{2k+1}$

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Yang-Yang Equation

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1st NDR

$\rho(\eta)$ $\longleftrightarrow$ $\sigma(\eta)$

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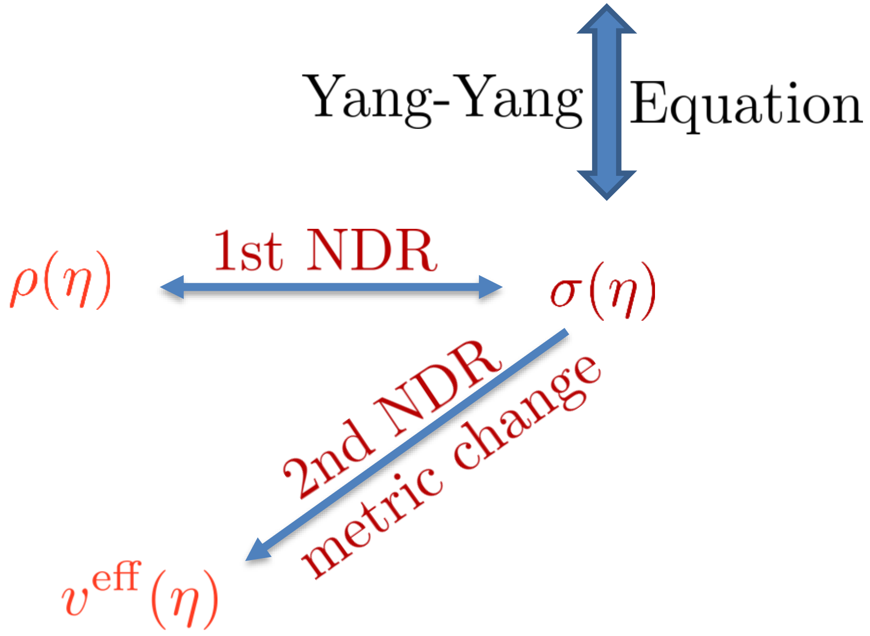

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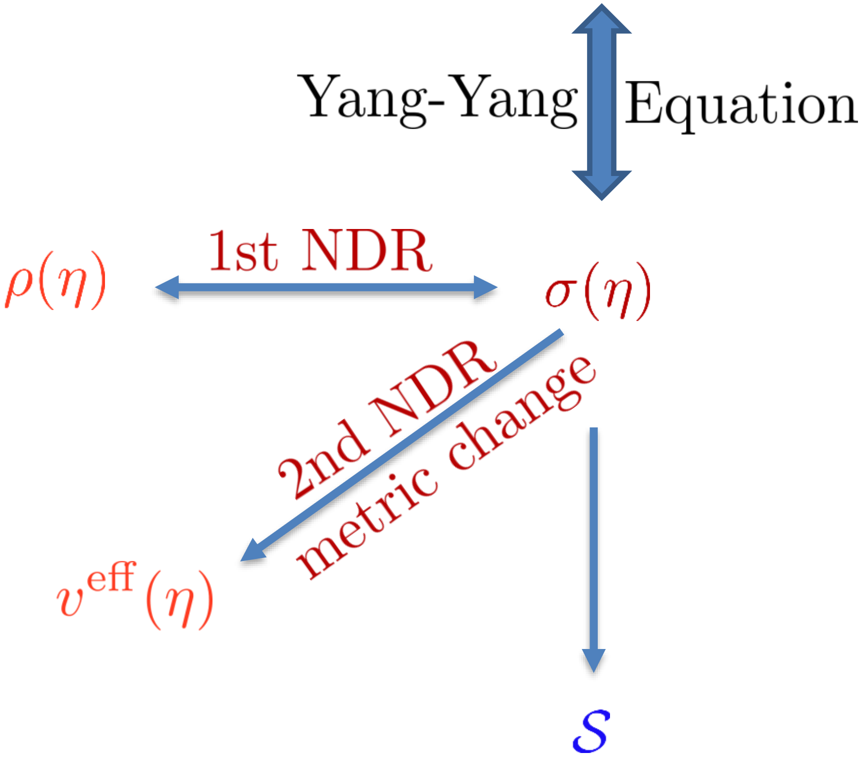

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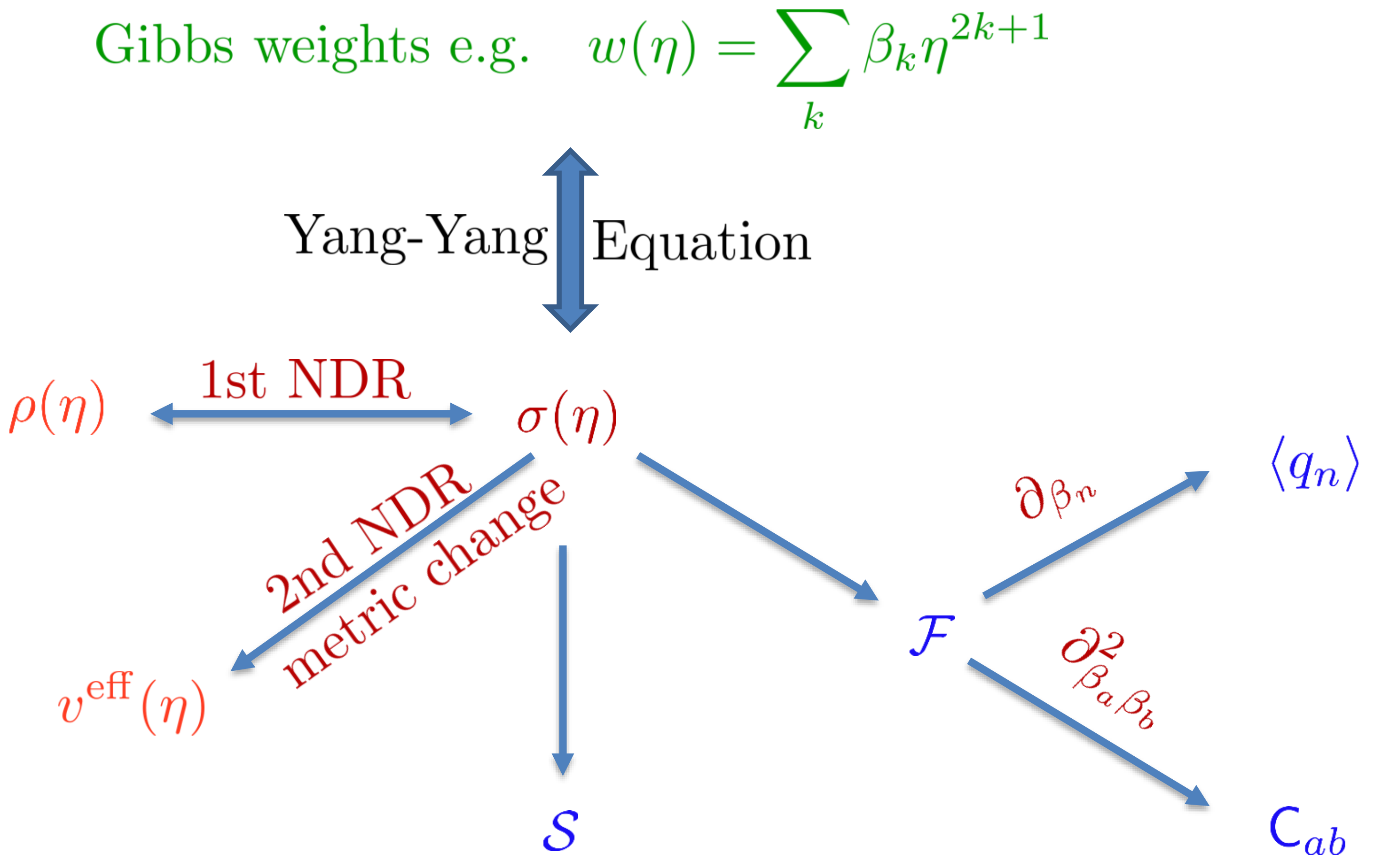
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Yang-Yang Equation
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1st NDR
$\sigma(\eta)$
2nd NDR
metric change
$v^{\mathrm{eff}}(\eta)$
$\mathcal{S}$
$\mathcal{F}$
∂_{β_n}
$\langle q_n \rangle$
$\partial^2_{\beta_a \beta_b}$
C_{ab}