Generalised hydrodynamics of the KdV soliton gas

Mini-colloque RNL 2023:
Turbulence intégrable, gaz de solitons, et hydrodynamique généralisée.

Thibault Bonnemain, 28th March 2023

[joint work with Benjamin Doyon and Gennady El]

KdV and solitons

- KdV: integrable nonlinear dispersive PDE

$$\partial_t \varphi + 6\varphi\partial_x\varphi + \partial_x^3\varphi = 0\ .$$

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- Infinite set of conservation laws

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"charges"

$$Q_n = \int \mathrm{d}x \ q_n(x,t) \ , \qquad J_n = \int \mathrm{d}t \ j_n(x,t) \ ,$$

Space
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$$\boxed{\partial_t q_n + \partial_x j_n = 0 \ .}$$

[Miura, Gardner, Kruskal (1968)]

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- Solvable via Inverse Scattering Transfom.

[Gardner, Greene, Kruskal, Miura (1967)]

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- Multisoliton solution

$$\varphi_N \sim \sum_{i=1}^{N} 2\eta_i^2 \mathrm{sech}^2 \left[\eta_i \left(x - 4\eta_i^2 t - x_i^{\pm}\right)\right] \quad \text{as} \quad t \to \pm\infty.$$

[Zakharov (1971)]

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- Relation between asymptotic states given by scattering shift

$$x_i^+ - x_i^- = \sum_j \frac{\operatorname{sgn}(\eta_\mathrm{i} - \eta_\mathrm{j})}{\eta_i} \ln \left| \frac{\eta_i + \eta_j}{\eta_i - \eta_j} \right|$$

[Lax (1968)]

Thermodynamics

- N-soliton partition function

$$\mathcal{Z}_N = \frac{1}{N!} \int_{\Gamma^N \times \mathbb{R}^N} \prod_{i=1}^{N} \frac{\mathrm{d}p(\eta_i)}{2\pi} \mathrm{d}x_i^- \exp\left[-\sum_{i=1}^{N} w(\eta_i)\right] \chi\left(\varphi_N(x, t=0) < \epsilon_x, x \notin [0, L]\right)$$

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Soliton bare velocity

$$p(\eta) = 4\eta^2$$

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$$\text{e.g.} \quad w(\eta) = \sum_k \beta_k h_k(\eta)$$

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- Asymptotic constraint

$$\int_{\mathbb{R}^N}\prod_{i=1}^{N}\mathrm{d}x_i^-\,\chi\left(\varphi_N(x,t=0),x\notin[0,L]\right)\approx\prod_{i=1}^{N}\left(\int_{x_i^{\mathrm{left}}}^{x_i^{\mathrm{right}}}\mathrm{d}x^-\right)=\prod_{i=1}^{N}L_N^i$$

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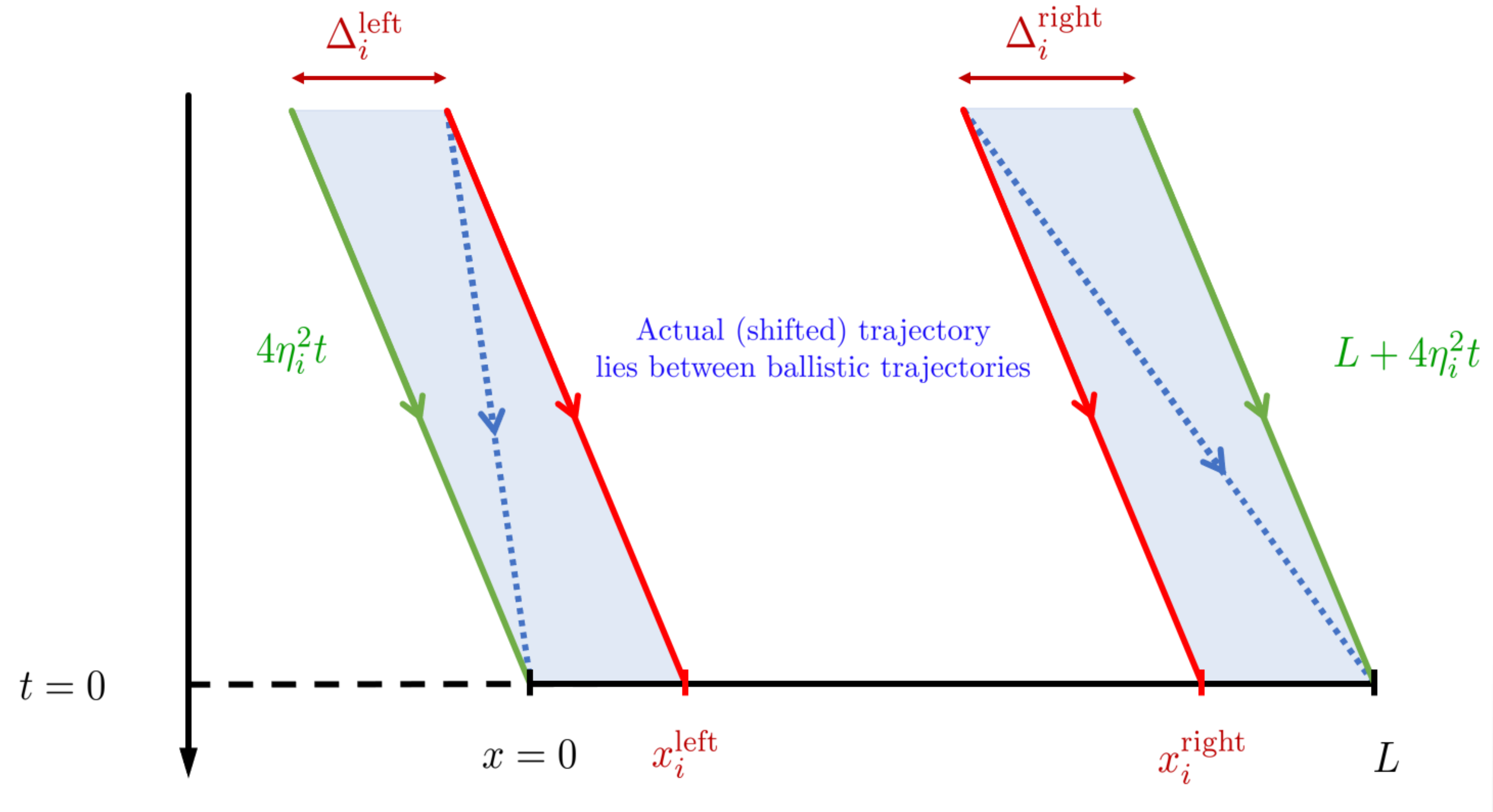
$$\text{with}\quad L_N^i = L\left(1-\frac{1}{L\eta_i}\sum_{j\neq i}^{N}\log\left|\frac{\eta+\mu}{\eta-\mu}\right|\right)$$

BA inspired approach

Asymptotic space



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$$\underbrace{0}_{\text{Position at t=0}} = \overbrace{x_i^{\text{left}}}^{\text{Asymptotic position}} - \underbrace{\frac{1}{\eta_i} \sum_{j \in T_{\text{left}}} \log \left| \frac{\eta_i + \eta_j}{\eta_i - \eta_j} \right|}_{\text{Shifts from faster solitons}} .$$

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Asymptotic space is shorter than real space

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constraint contributes to : $L^N \prod_{i=1}^{N} \left(1 - \frac{1}{L\eta_i} \sum_{j \neq i}^{N} \log \left| \frac{\eta + \mu}{\eta - \mu} \right| \right)$

Asymptotic space density

- Let $L_N(\eta)$ interpolate L_i

$$\mathcal{K}_N(\eta) \equiv \frac{L_N(\eta)}{L} = 1 - \frac{1}{L\eta} \sum_j \log \left| \frac{\eta + \eta_j}{\eta - \eta_j} \right| \ .$$

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Aymptotic space density

$$\mathrm{d}x^-(\eta) = \mathcal{K}(\eta)\mathrm{d}x$$

change of metric due to interactions

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- Same structure as NDR of soliton gas theory

[El (2003)]

$$\sigma(\eta)u(\eta) = \eta - \int_\Gamma \mathrm{d}\mu \ u(\mu) \log\left|\frac{\eta+\mu}{\eta-\mu}\right|$$

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$$\mathcal{K}(\eta) \leftrightarrow \frac{\sigma(\eta)}{\eta}u(\eta)$$

$$u(\eta) \leftrightarrow \rho(\eta)$$

Thermodynamic equilibrium

- Large deviations theory

$$\mathcal{Z}_N \asymp \exp\left(-L\mathcal{F}^{\mathrm{MF}}[u(\eta)]\right)$$

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$$\underbrace{-\log\left(1 - \frac{1}{\eta}\int_\Gamma \mathrm{d}\mu\ \rho(\eta)\log\left|\frac{\eta+\mu}{\eta-\mu}\right|\right)}_{\text{Constraint}} \underbrace{-1 + \log\rho(\eta)}_{\text{Configuration entropy}}\Bigg]$$

[Sanov (1961)]

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- Minimisation condition (Yang-Yang equation)

$$0 = \left.\frac{\delta\mathcal{F}^{\mathrm{MF}}[\rho]}{\delta\rho(\eta)}\right|_{\rho=u} \quad \Rightarrow \quad \log\frac{4\sigma(\eta)}{\pi} = w(\eta) + \int_\Gamma \frac{\mathrm{d}\mu}{\sigma(\mu)}\log\left|\frac{\eta+\mu}{\eta-\mu}\right|$$

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- Entropy

$$\mathcal{S} = \underbrace{\mathcal{W}}_{\int_\Gamma \mathrm{d}\eta\; u(\eta)w(\eta)} - \mathcal{F} = \int_\Gamma \mathrm{d}\mu\; u(\mu)\left(1 + \log\frac{4\sigma(\eta)}{\pi}\right)$$

Thermodynamic equilibrium (GHD formalism)

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$$\epsilon(\eta) = w(\eta) - \int \frac{\mathrm{d}p(\mu)}{2\pi\mu} \log\left|\frac{\eta+\mu}{\eta-\mu}\right| F(\epsilon(\mu))$$

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$$F \equiv -e^{-\epsilon(\eta)}$$

free energy density

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occupation function

↓

Maxwell-Boltzmann

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⇨ $$n(\eta) = \left.\frac{dF}{d\epsilon}\right|_{\epsilon=\epsilon(\eta)} = e^{-\epsilon(\eta)} = \frac{\pi}{4\sigma(\eta)} \propto \frac{u(\eta)}{\eta\mathcal{K}(\eta)}$$

occupation function

Maxwell-Boltzmann

Density of solitons in the asymptotic space

Some more (maybe) familiar relations

- Density of states

$$u(\eta) = \frac{\delta \mathcal{F}}{\delta w(\eta)}$$

- Thermodynamic averages

$$\langle q_n \rangle = \int_\Gamma \mathrm{d}\eta \; u(\eta) h_n(\eta) = \frac{\partial \mathcal{F}}{\partial \beta_n}$$

- Correlations

$$\mathsf{C}_{ab} \equiv \int_\Gamma \mathrm{d}x \left(\langle q_a(x) q_b(0) \rangle - \langle q_a(x) \rangle \langle q_b(0) \rangle \right) = -\frac{\partial^2 \mathcal{F}}{\partial \beta_a \partial \beta_b}$$

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[Doyon (2018)]

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[Doyon (2018)]

$$h^{\mathrm{dr}}(\eta) = h(\eta) + \int_\Gamma \frac{\mathrm{d}\mu}{\sigma(\mu)} \log \left| \frac{\eta - \mu}{\eta + \mu} \right| h^{\mathrm{dr}}(\mu)$$

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$$\langle q_n \rangle = \int_\Gamma \mathrm{d}\eta \; u(\eta) h_n(\eta) = \frac{\partial \mathcal{F}}{\partial \beta_n}$$

- Correlations

$$\mathsf{C}_{ab} \equiv \int_\Gamma \mathrm{d}x \left(\langle q_a(x) q_b(0) \rangle - \langle q_a(x) \rangle \langle q_b(0) \rangle \right) = -\frac{\partial^2 \mathcal{F}}{\partial \beta_a \partial \beta_b}$$

	MB	FD	BE	Simulations
$\mathsf{C}_{00}^{\mathrm{DC}}$	0.0235	-2.28	2.32	0.022 ± 0.003
$\mathsf{C}_{01}^{\mathrm{DC}}$	0.027	-3.18	3.23	0.024 ± 0.004
$\mathsf{C}_{11}^{\mathrm{DC}}$	0.042	-4.48	4.56	0.039 ± 0.005
$\mathsf{C}_{00}^{\mathrm{U}}$	0.22	0.028	0.41	0.2 ± 0.03
$\mathsf{C}_{01}^{\mathrm{U}}$	0.28	0.072	0.49	0.23 ± 0.04
$\mathsf{C}_{11}^{\mathrm{U}}$	0.39	0.12	0.66	0.36 ± 0.05

Recap

Gibbs weights e.g. $w(\eta) = \sum_k \beta_k \eta^{2k+1}$

Recap

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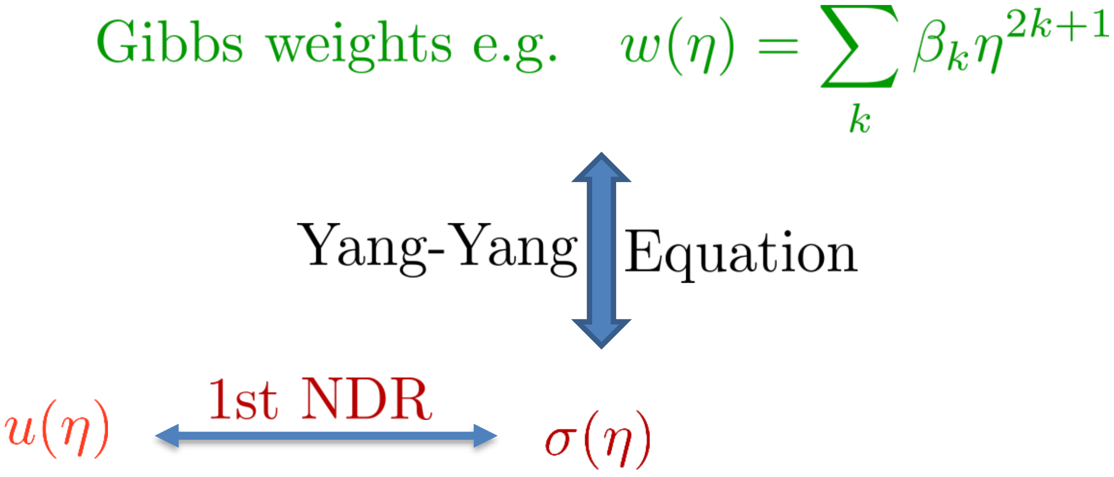
Yang-Yang $\Updownarrow$ Equation

$\sigma(\eta)$

Recap

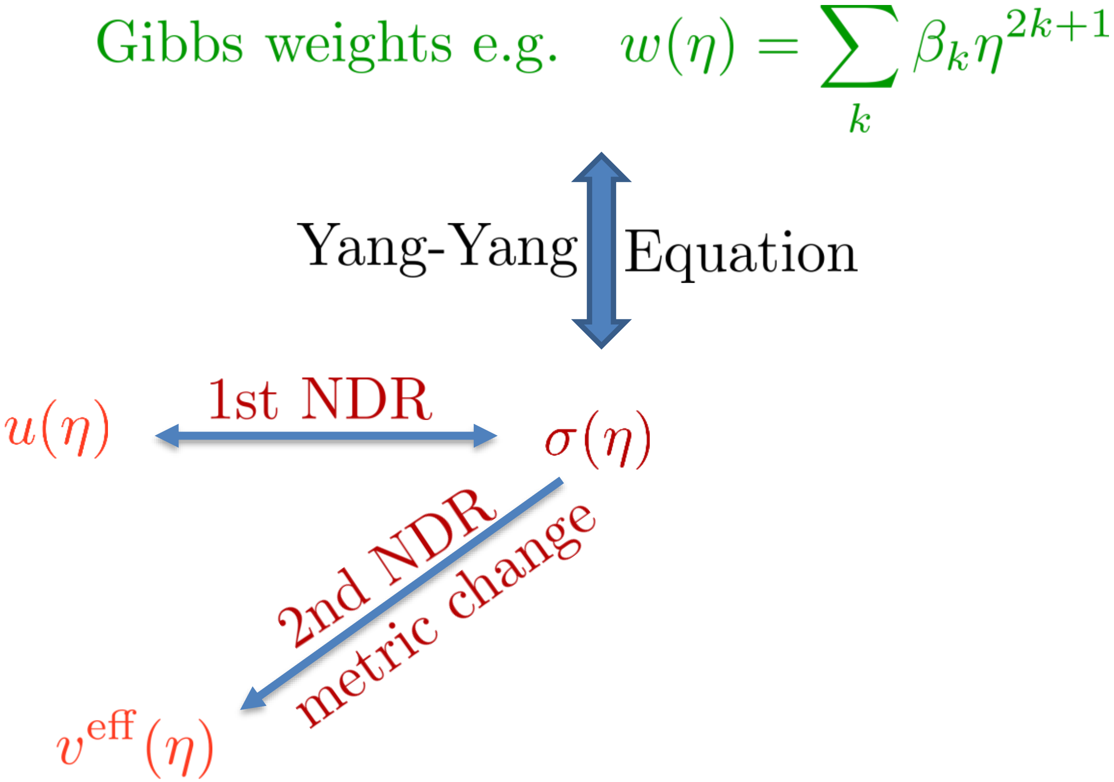


Gibbs weights e.g. $w(\eta) = \sum_k \beta_k \eta^{2k+1}$

Yang-Yang Equation

$u(\eta)$ 1st NDR $\sigma(\eta)$

Recap



Gibbs weights e.g. $w(\eta) = \sum_k \beta_k \eta^{2k+1}$

Yang-Yang Equation

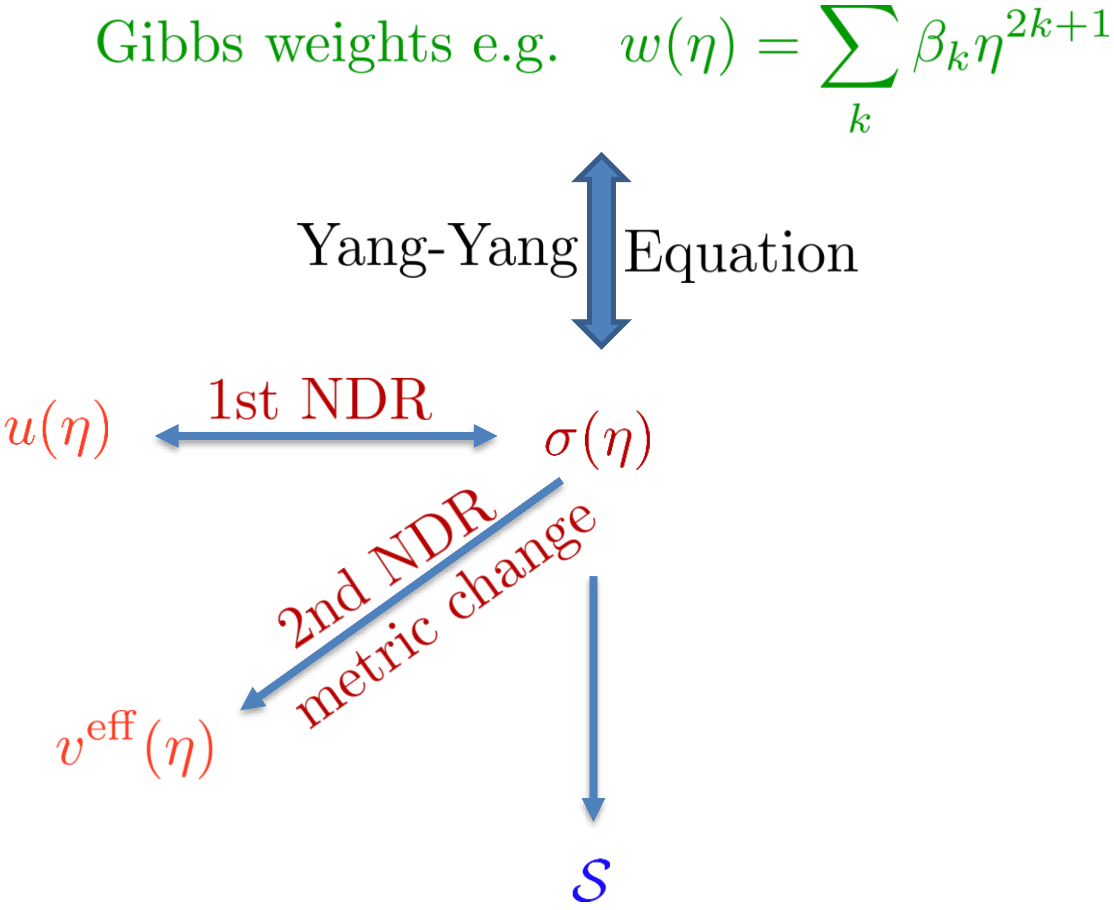
$u(\eta)$ 1st NDR $\sigma(\eta)$

2nd NDR

metric change

$v^{\mathrm{eff}}(\eta)$

Recap



Gibbs weights e.g. $w(\eta) = \sum_k \beta_k \eta^{2k+1}$

Yang-Yang Equation

$u(\eta)$

1st NDR

$\sigma(\eta)$

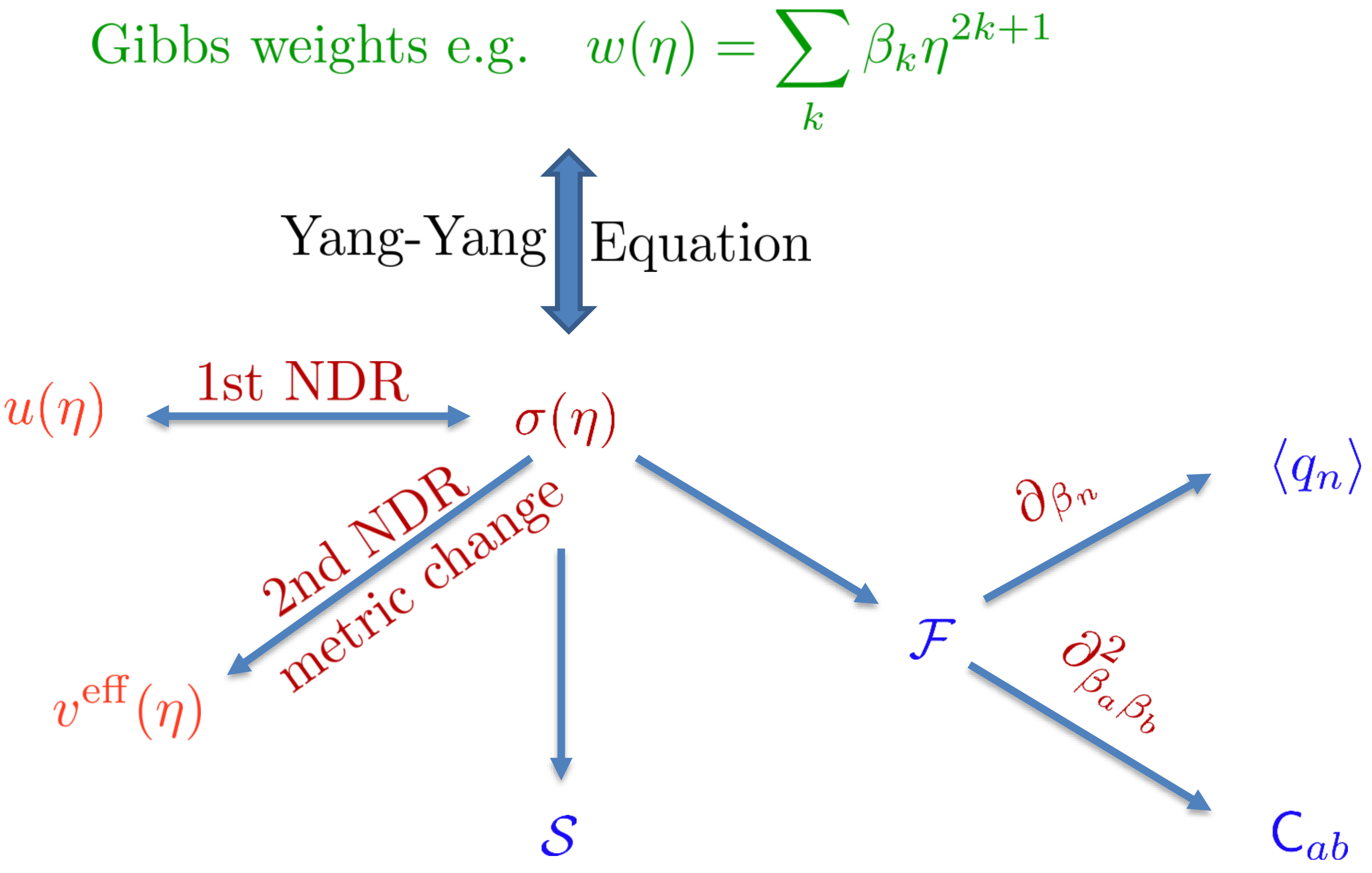
2nd NDR

metric change

$v^{\text{eff}}(\eta)$

$\mathcal{S}$

Recap



$$\begin{tikzcd}
 & & \text{Gibbs weights e.g.}\quad w(\eta)=\sum_k \beta_k \eta^{2k+1} & & \\
u(\eta) \arrow[rr, leftrightarrow, "\text{1st NDR}"] & & \sigma(\eta) \arrow[u, leftrightarrow, "\text{Yang-Yang Equation}"] \arrow[dl, "\text{2nd NDR}"', "\text{metric change}"] \arrow[dd] \arrow[dr] & & \langle q_n \rangle \\
 & v^{\text{eff}}(\eta) & & \mathcal{F} \arrow[ur, "\partial_{\beta_n}"] \arrow[dr, "\partial^2_{\beta_a \beta_b}"] & \\
 & & \mathcal{S} & & \mathsf{C}_{ab}
\end{tikzcd}$$