

Quadratic Mean Field Games: Schrödinger and electrostatic representations

Northumbria University

Thibault Bonnemain, 8th March 2023

[Based on works w/ D. Ullmo, T. Gobron, M. Butano, C. Appert-Rolland, I. Echeverria-Huarte A. Nicolas, A. Seguin]

Game theory

Def: Mathematical framework to study strategic optimization.

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Hawk-Dove paradigm

	H	D
H	$(\frac{1}{4}; \frac{1}{4})$	$(1; 0)$
D	$(0; 1)$	$(\frac{1}{2}; \frac{1}{2})$

(3)

(2)

Mean Field Games

- Subdiscipline of Game Theory: problems of optimization with interacting agents in the large N limit.
- Relatively recent: seminal papers published in 2006.

[Lasry, Lions (2006)]

[Huang, Malhamé, Caines (2006)]

- Wide literature: mathematics, engineering sciences, economics, sociology ...

[Lachapelle, Wolfram (2011)]

[Guéant et al. (2012)]

[Gomes, Saude (2014)]

[Laguzet, Turinici (2015)]

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[Bonnemain et al. (2022)]

Diapositive 4

tb20 thibault bonnemain; 18/02/2020

tb21 Lachap + bonne pedestrian dynamics
Guéant + cardia finance
Achdou mining industries
Gomes early survey
thibault bonnemain; 03/11/2022

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Physicist approach: focus on models that can be understood completely.

Diapositive 5

tb23 thibault bonnemain; 18/02/2020

tb24 Lachap + bonne pedestrian dynamics
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MFG paradigm: population dynamics

- N agents $i = 1, 2, \dots, N$ ($N \gg 1$)

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- state of agent $i \rightarrow$ real vector \mathbf{X}^i (position, capital, beliefs...)

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- agent's dynamic

$$d\mathbf{X}_t^i = \mathbf{a}_t^i dt + \sigma d\mathbf{w}_t^i$$

$d\mathbf{w}_t^i \equiv$ white noise

drift $\mathbf{a}_t^i \equiv$ control parameter

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$$\mathbb{E} \left[\int_t^T \underbrace{\left(\frac{\mu}{2} (\mathbf{a}_s^i)^2 - V[m](\mathbf{X}_s^i, \mathbf{s}) \right)}_{\text{running cost}} ds + \underbrace{c_T(\mathbf{X}_T^i)}_{\text{terminal cost}} \right]$$

(9)

(4)

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Mean Field Games equations

- Optimization: linear programming leads to a (backward) Hamilton-Jacobi-Bellman equation for the value function $u(\mathbf{x}, t)$

$$\begin{cases} \partial_t u + \frac{1}{2\mu} (\nabla_{\mathbf{x}} u)^2 + \frac{\sigma^2}{2} \Delta_{\mathbf{x}} u = V[m](x, t) \\ u(x, t=T) = c_T(x) \end{cases} \quad (\text{HJB}) .$$

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- Langevin dynamic $d\mathbf{X}_t^i = \mathbf{a}_t^i dt + \sigma d\mathbf{w}_t^i$ leads to a (forward) diffusion equation for the density $m(x, t)$

$$\begin{cases} \partial_t m - \nabla_{\mathbf{x}} \left[m \frac{\nabla_{\mathbf{x}} u}{\mu} \right] - \frac{\sigma^2}{2} \Delta_{\mathbf{x}} m = 0 \\ m(x, t=0) = m_0(x) \end{cases} \quad (\text{Kolmogorov}) .$$

{ 4 }

{ 5 }

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optimal control \mathbf{a}

Mean Field Game = coupling between a (collective) stochastic motion and an (individual) optimization problem through a mean field

Ergodic state

Theorem:

[Cardaliaguet, Lasry, Lions, Poretta (2013)]

- $V[m](\mathbf{x}, t)$ has no explicit time dependence
- Long optimization time: $T \rightarrow \infty$
- System is confined
- ... + other conditions ...



$$\text{for } 0 \ll t \ll T \quad \left| \begin{array}{l} m(\mathbf{x}, t) \simeq m_e(\mathbf{x}) \\ u(\mathbf{x}, t) \simeq u_e(\mathbf{x}) + \lambda t \end{array} \right.$$

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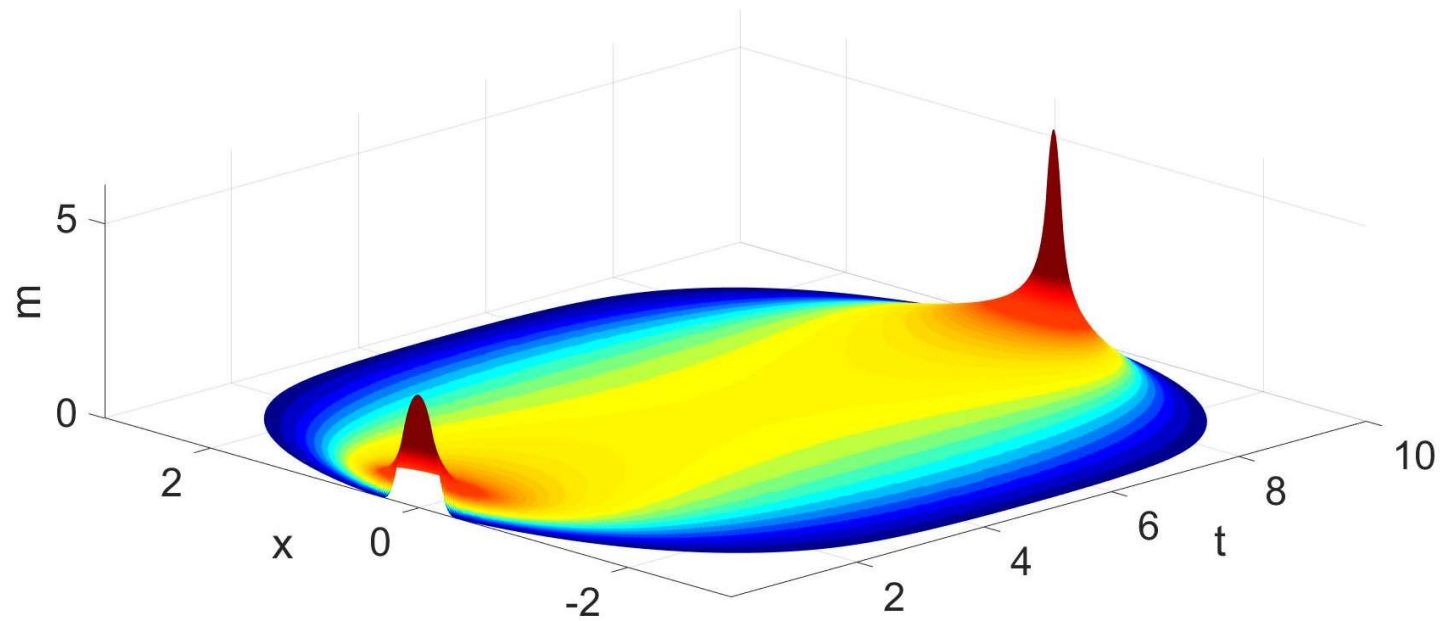
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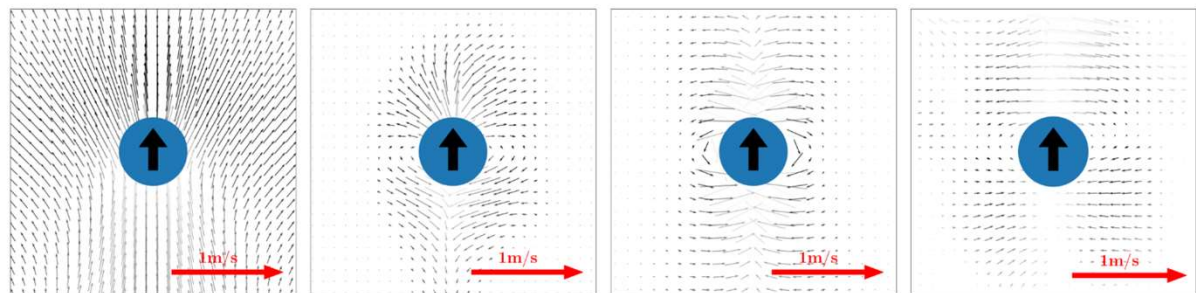
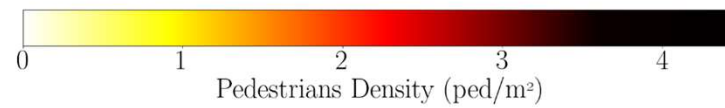
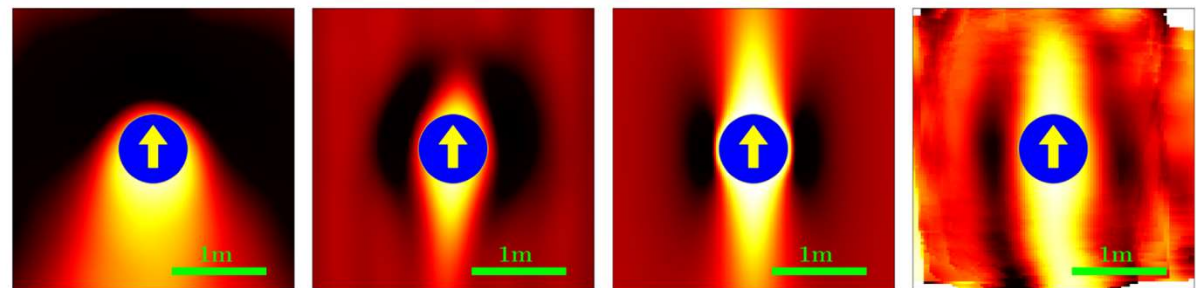
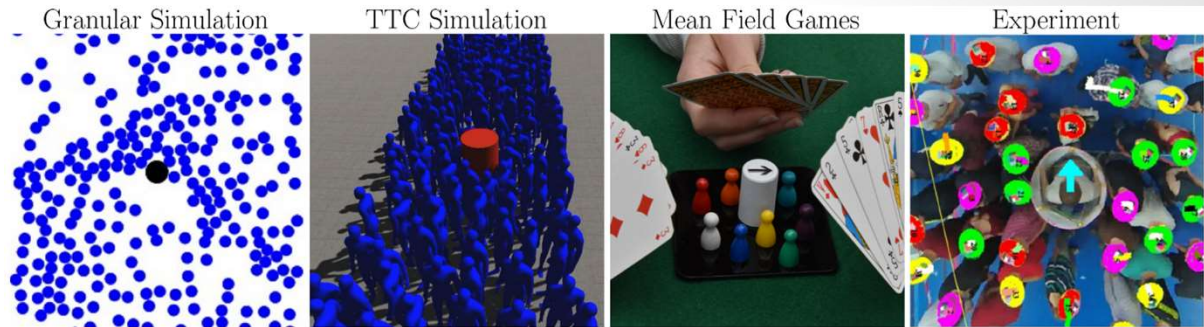
$$(m_e, u_e, \lambda) \text{ such that } \begin{cases} \lambda - \frac{1}{2\mu} (\nabla_{\mathbf{x}} u_e)^2 + \frac{\sigma^2}{2} \Delta_{\mathbf{x}} u_e = V[m_e](\mathbf{x}) \\ \nabla_{\mathbf{x}}(m_e(\nabla_{\mathbf{x}} u_e)) - \frac{\sigma^2}{2} \Delta_{\mathbf{x}} m_e = 0 \end{cases}$$

A simple example



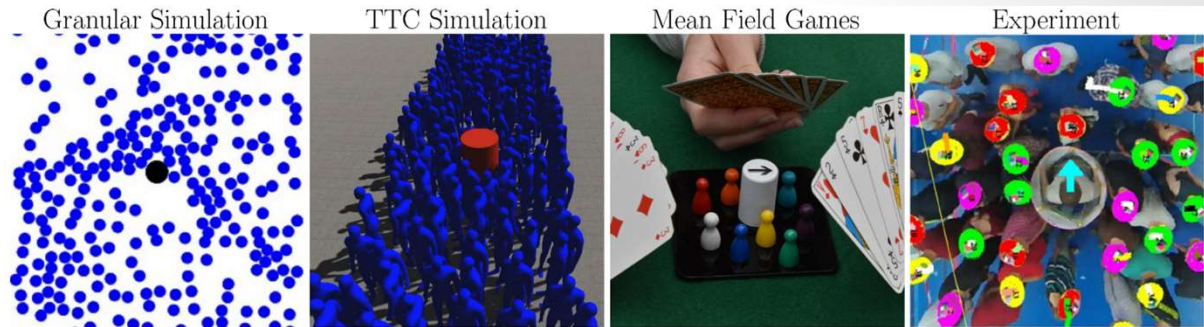
“Search party” toy model: $V[m](x, t) = \underbrace{gm(x, t)}_{< 0} + \underbrace{U_0(x)}_{\propto -x^2}$

Applications to Crowd dynamics

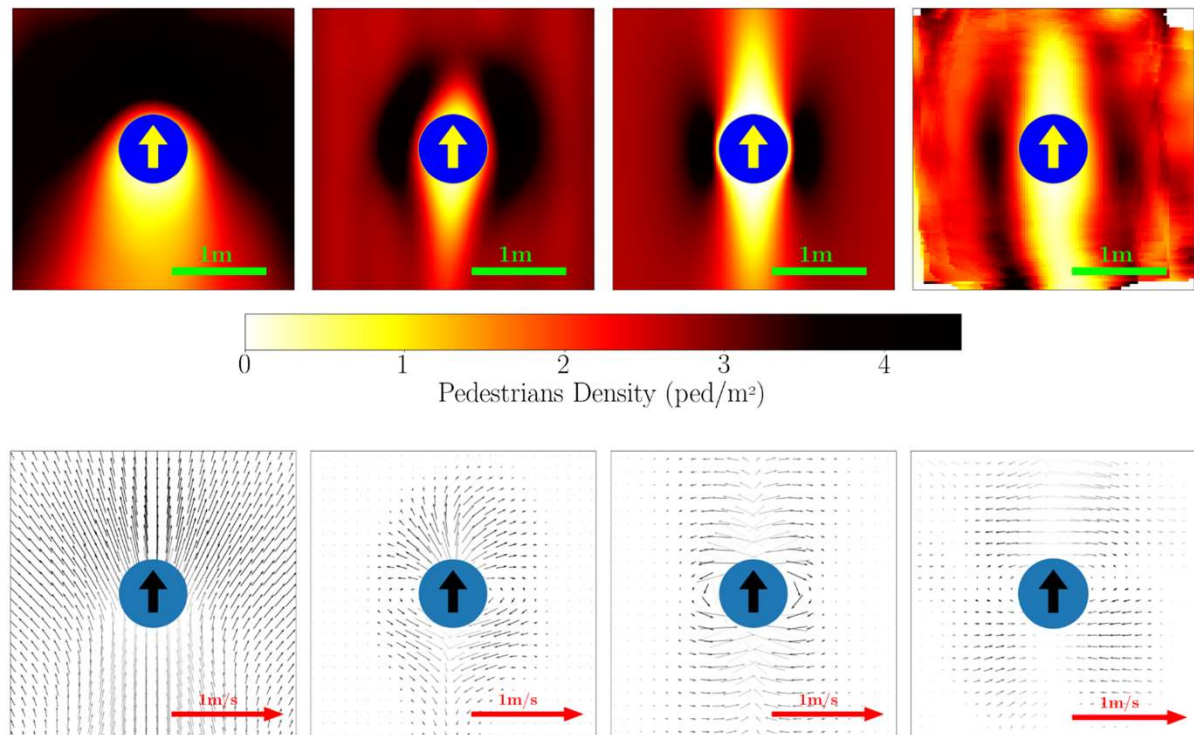


[Bonnemain et al. PRE (2023)]

Applications to Crowd dynamics

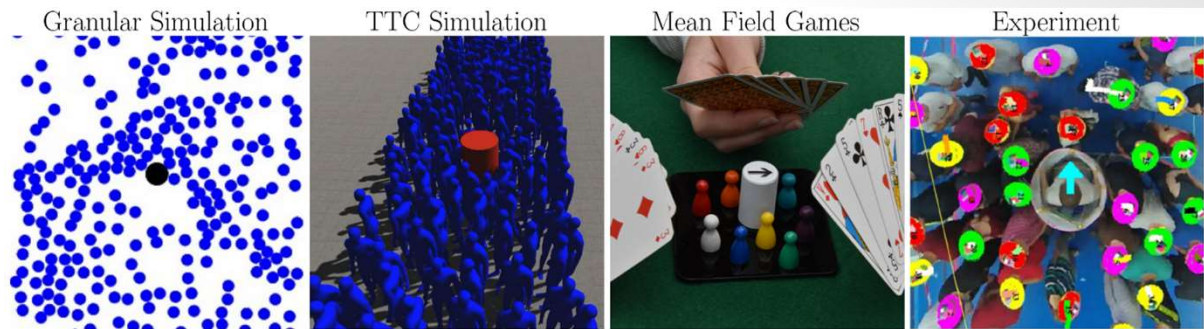


Symmetric density

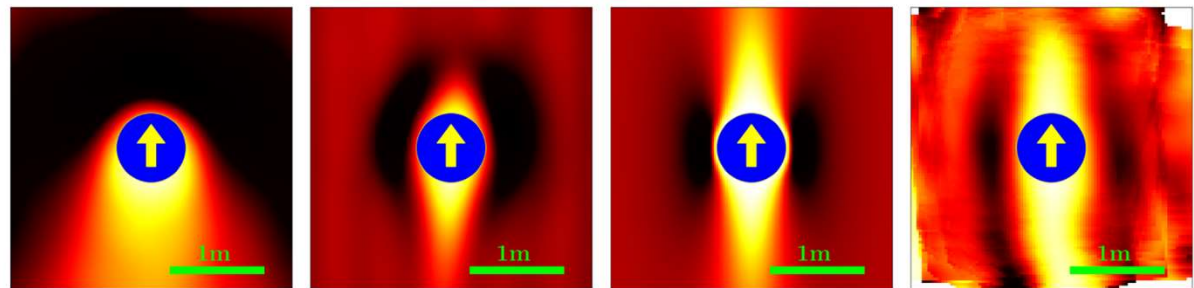


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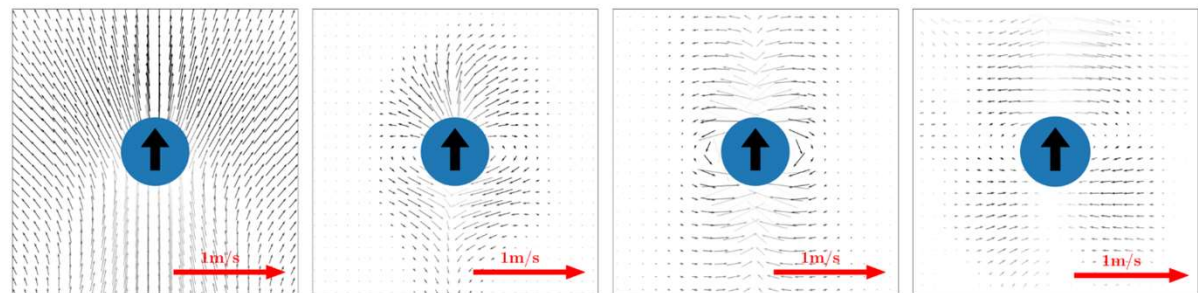
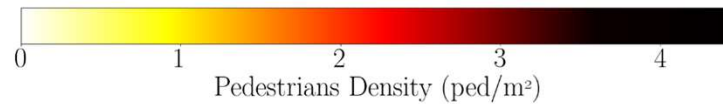
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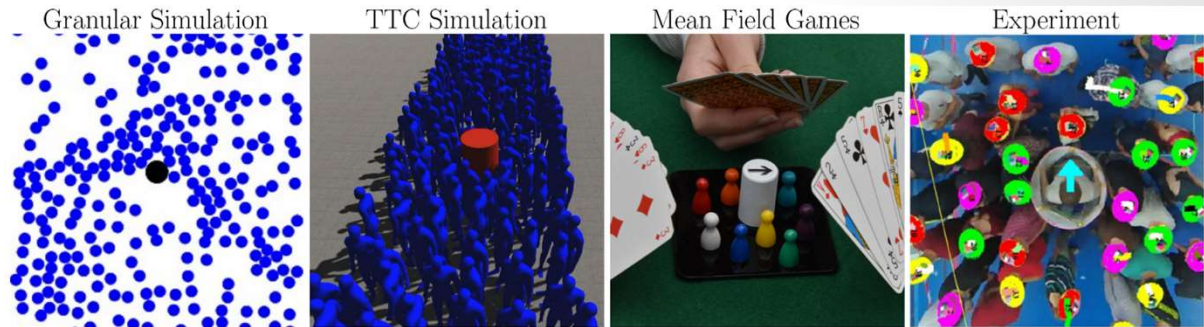


Transverse displacement



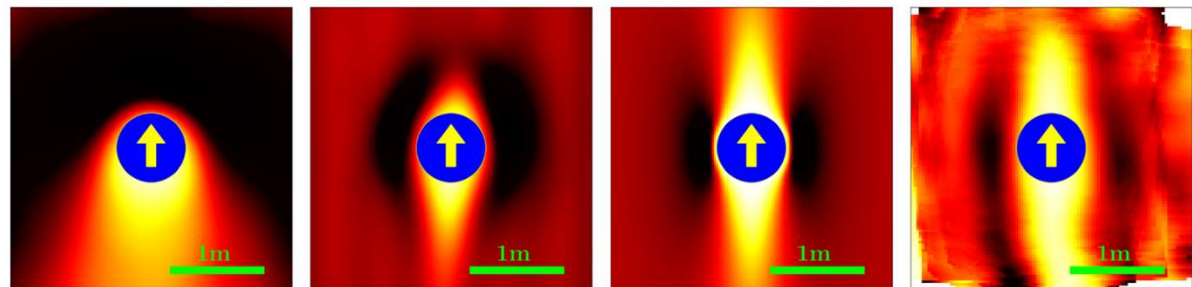
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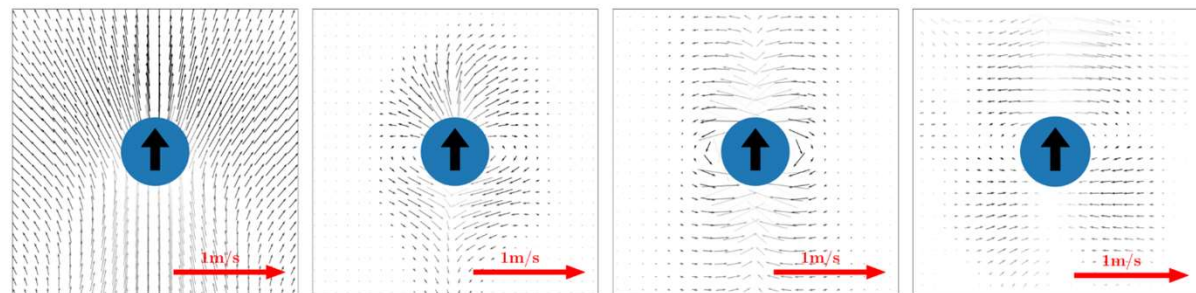
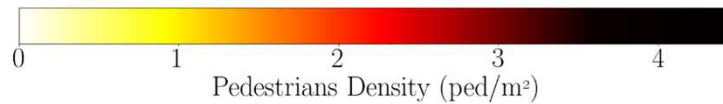


Symmetric density

Only reproduced
by MFG



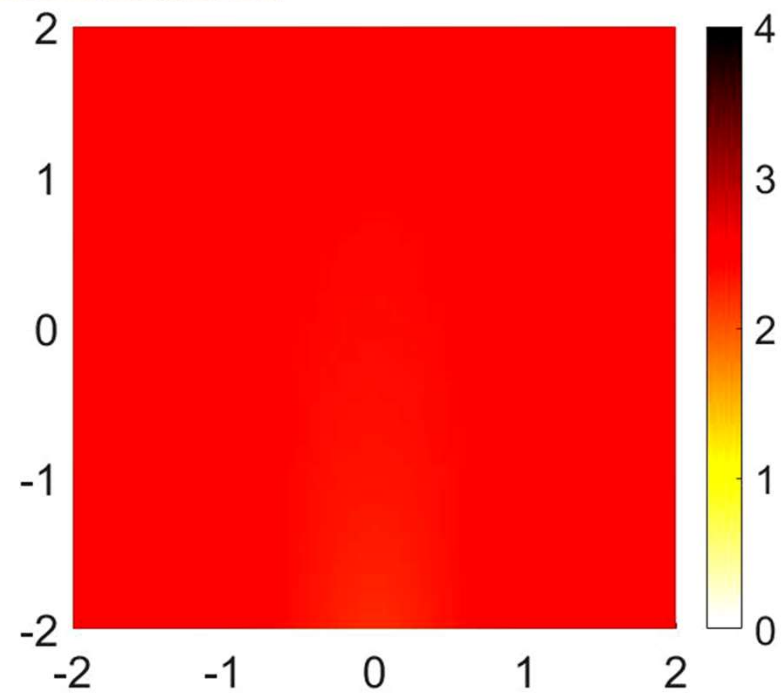
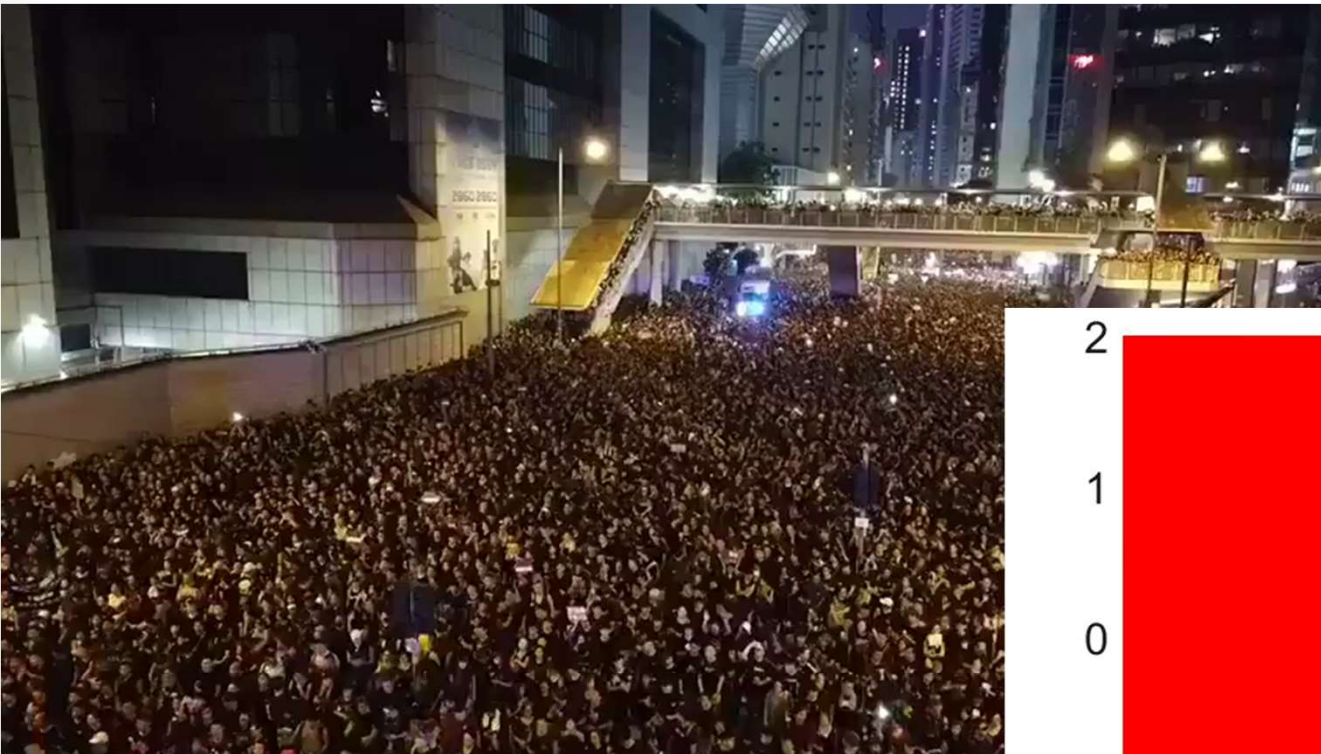
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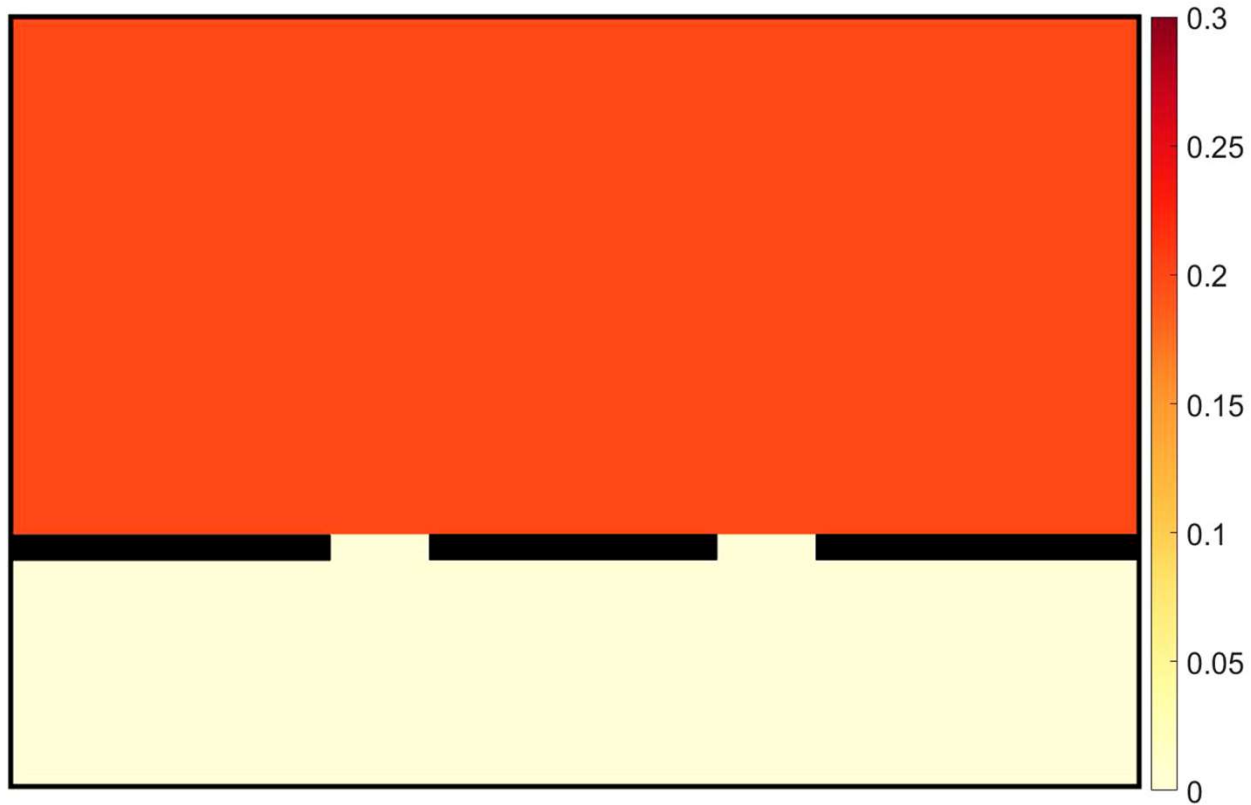
Qualitative examples

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Non-linear Schrödinger representation

[Bonnemain, Gobron, Ullmo Phys.Lett. A (2020) ; SciPost (2020) ; J. Math. Phys. (2021)]

Introduce two new variables $\Phi(x, t)$, $\Gamma(x, t)$ defined by:

$$u(x, t) = -\mu\sigma^2 \log(\Phi(x, t)) \quad m(x, t) = \Gamma(x, t)\Phi(x, t)$$

$$\Rightarrow \begin{cases} \mu\sigma^2 \partial_t \Gamma = \frac{\mu\sigma^4}{2} \Delta_{\mathbf{x}} \Gamma + U_0(\mathbf{x})\Gamma + g m \Gamma \\ -\mu\sigma^2 \partial_t \Phi = \frac{\mu\sigma^4}{2} \Delta_{\mathbf{x}} \Phi + U_0(\mathbf{x})\Phi + g m \Phi \end{cases} \quad m = \Gamma\Phi$$

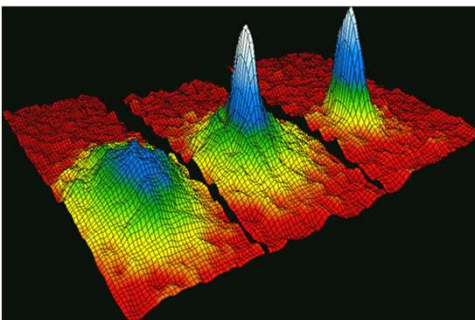
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Rubidium atoms (170 nK)

$$i\hbar\partial_t \Psi = -\frac{\hbar^2}{2\mu} \Delta_{\mathbf{x}} \Psi + U_0(\mathbf{x})\Psi + g|\Psi|^2 \Psi$$

Non-Linear Schrödinger

$$(\Psi, \Psi^*, \hbar) \rightarrow (\Phi, \Gamma, i\mu\sigma^2)$$

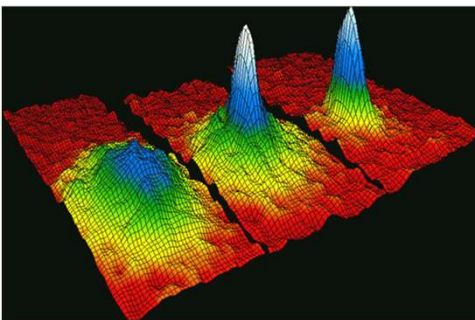
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Energy Conservation

- Quadratic MFG are variational systems

$$S[\Gamma, \Phi] \equiv \int_0^T dt \int_{\mathbb{R}} dx \left[\frac{\mu\sigma^2}{2} (\Gamma \partial_t \Phi - \Phi \partial_t \Gamma) \right. \\ \left. - \frac{\mu\sigma^4}{2} \nabla \Gamma \cdot \nabla \Phi + \left[U_0 + \frac{g}{2} \Gamma \Phi \right] \Gamma \Phi \right]$$

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Energy Conservation

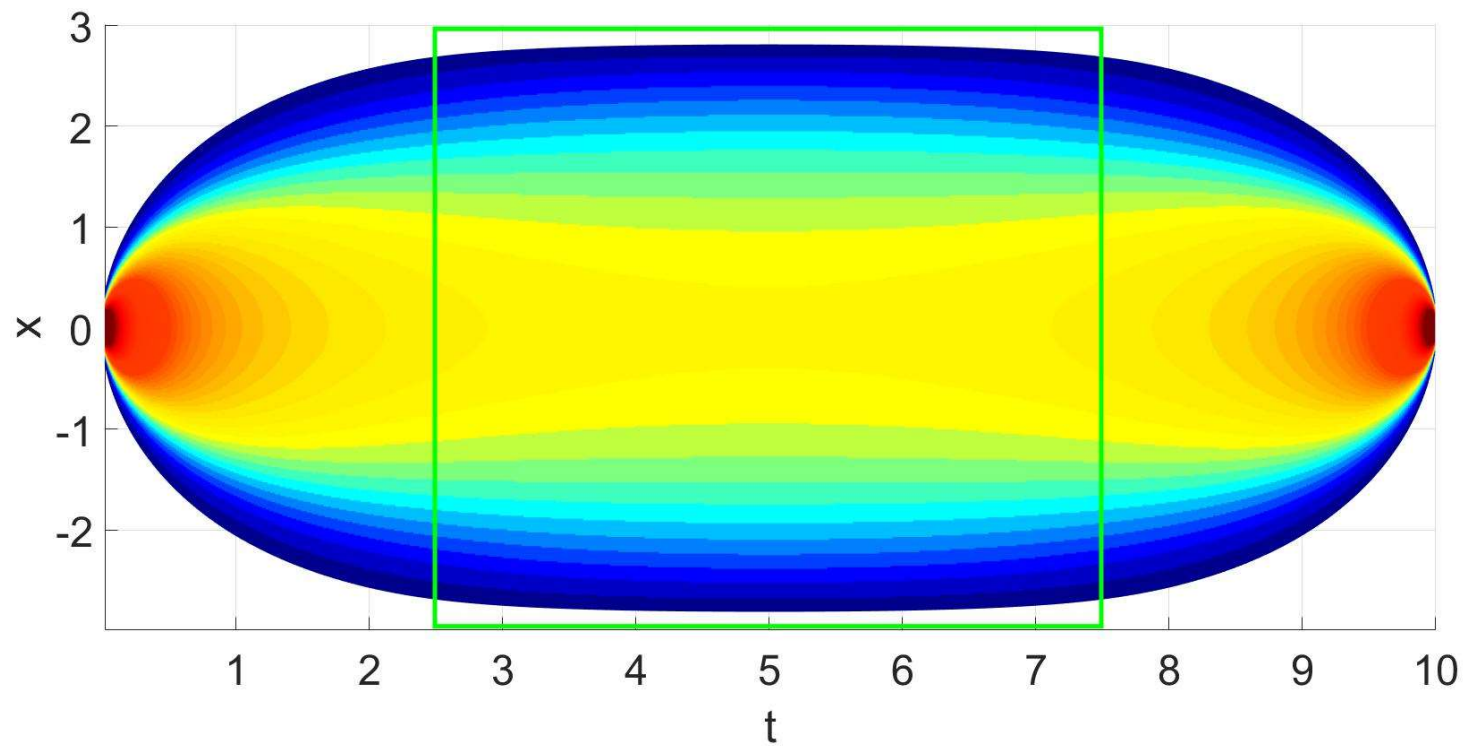
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$$\Rightarrow \frac{E^{\text{kin}}}{E^{\text{int}}} \sim \frac{\nu}{\Sigma} \quad \nu = \frac{\mu\sigma^4}{|g|}$$

The Ergodic State



Thomas-Fermi approximation

Ergodic MFG: Schrödinger representation

$$\lambda \Psi_e = \frac{\mu \sigma^4}{2} \Delta_x \Psi_e + U_0(x) \Psi_e + g |\Psi_e|^2 \Psi_e$$

Thomas-Fermi approximation

Limiting case: kinetic energy is negligible in front of interactions

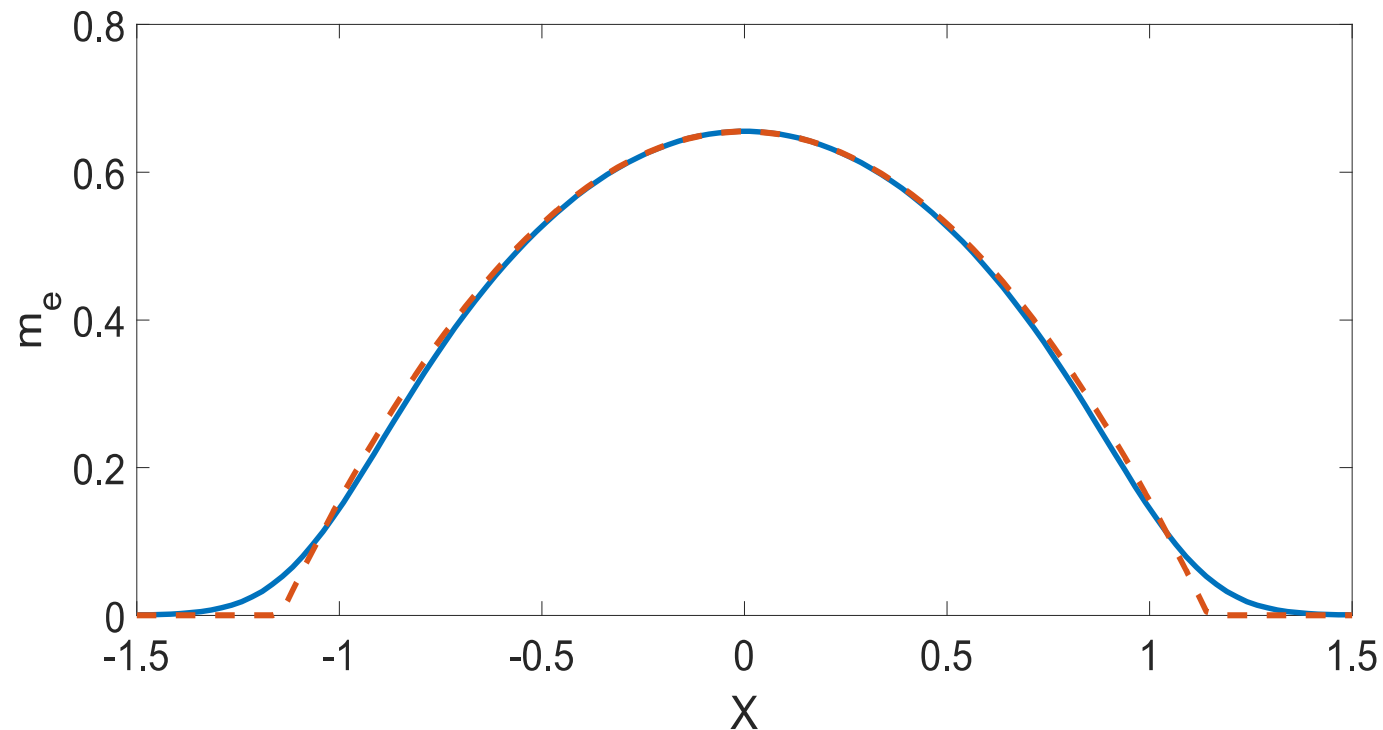
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$$\Psi_e = \begin{cases} \left(\frac{\lambda - U_0(x)}{|g|} \right)^{1/2} & \text{if } \lambda > U_0(x) \\ 0, & \text{otherwise} \end{cases}$$

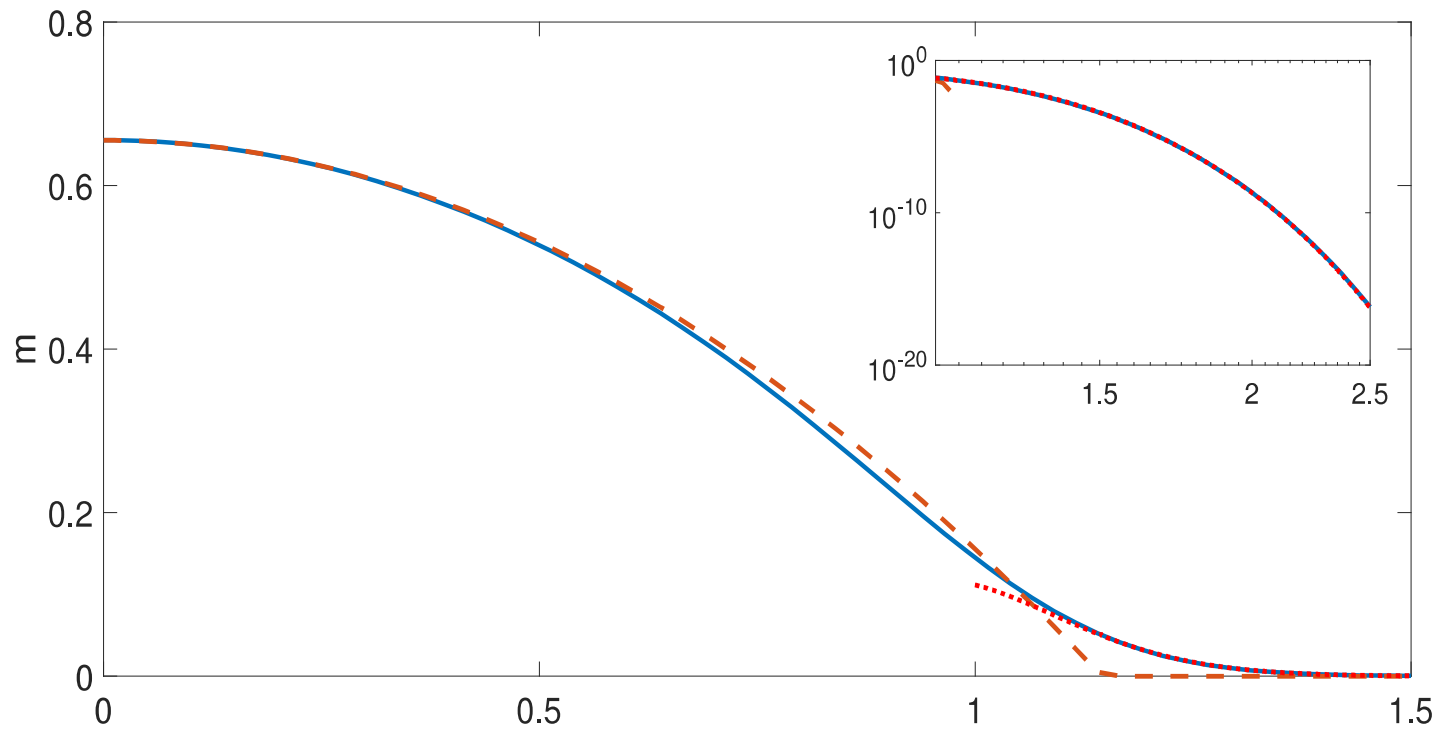
- Ψ_e represents either Φ_e or Γ_e
- $m_e = \Psi_e^2$
- λ is computed through the normalization of m_e

Thomas-Fermi approximation



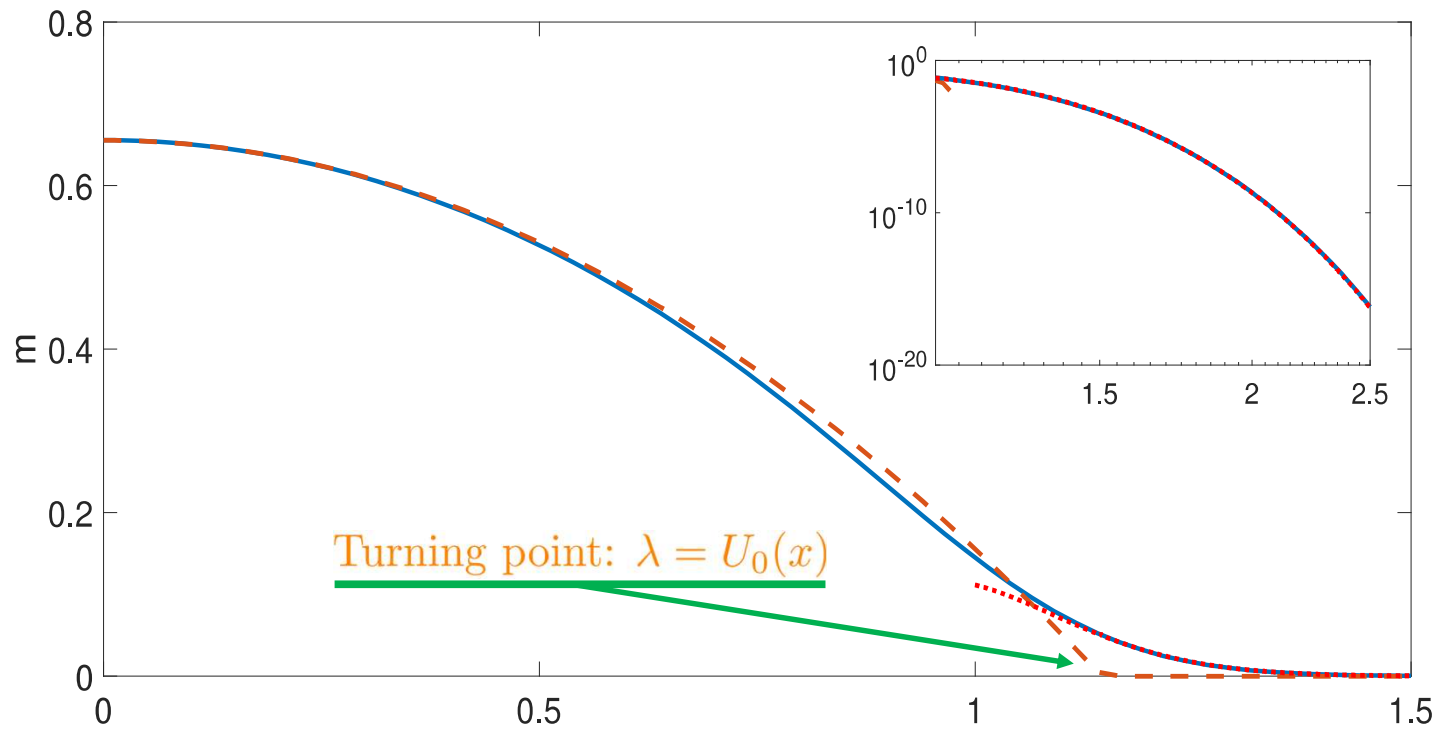
$$m_e = \begin{cases} \left(\frac{\lambda - U_0(x)}{|g|} \right) & \text{if } \lambda > U_0(x) \\ 0, & \text{otherwise} \end{cases} \quad \lambda = \left(\frac{3|g|\sqrt{\mu\omega_0^2}}{4\sqrt{2}} \right)^{2/3}$$

Semiclassical approximation in the tails



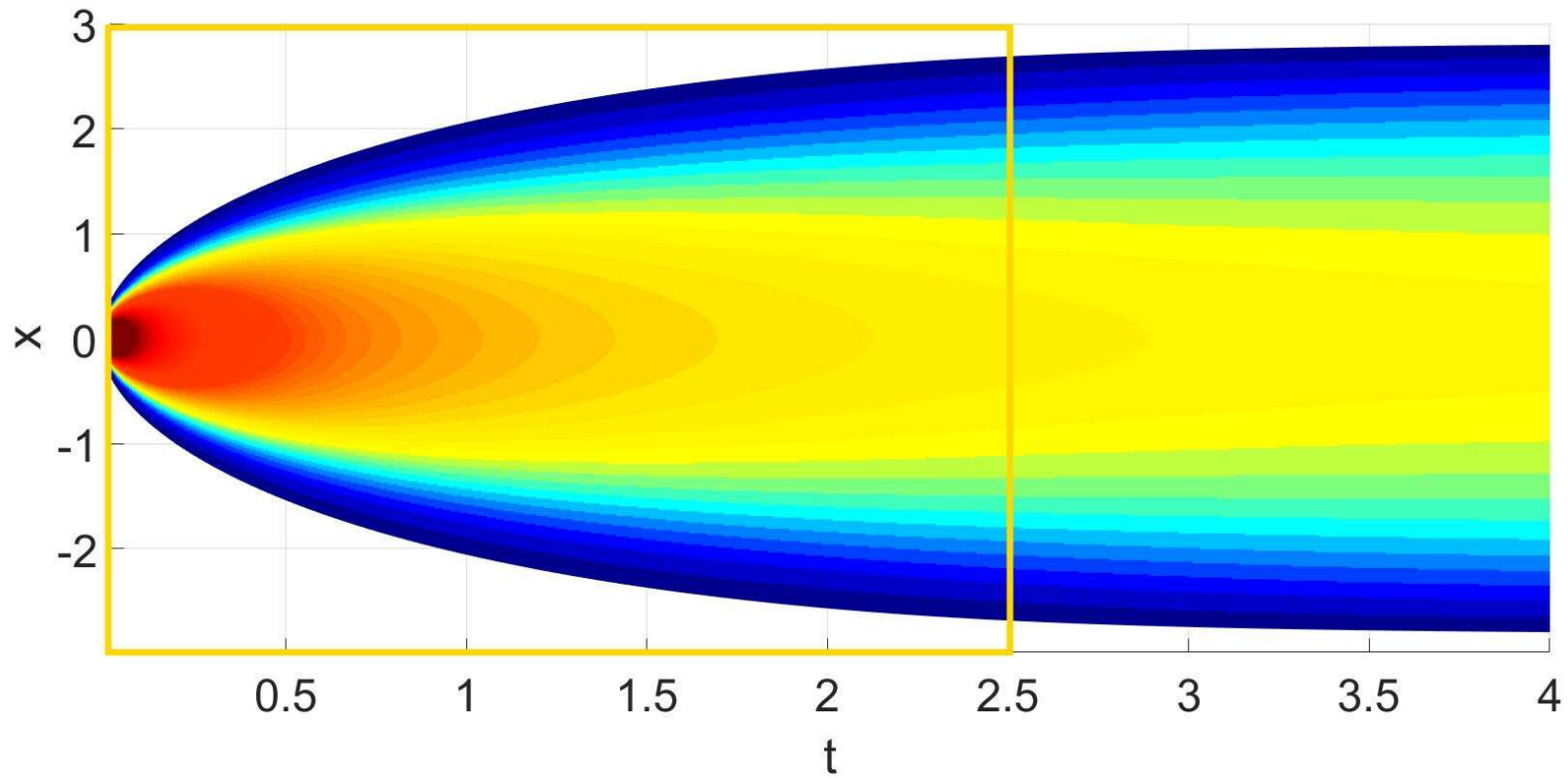
$$\Psi_{\text{SC}}(x) = \begin{cases} C_{\text{left}} \left(\frac{8\pi S_{\text{left}}(x)}{3U_0} \right)^{1/2} \cos\left(\frac{\pi}{3}\right) [J_{1/3}(S_{\text{left}}(x)) + J_{-1/3}(S_{\text{left}}(x))] \\ 2C_{\text{right}} \left(\frac{8S_{\text{right}}(x)}{\pi |U_0|} \right)^{1/2} \cos\left(\frac{\pi}{3}\right) K_{1/3}(S_{\text{right}}(x)) \end{cases}$$

Semiclassical approximation in the tails

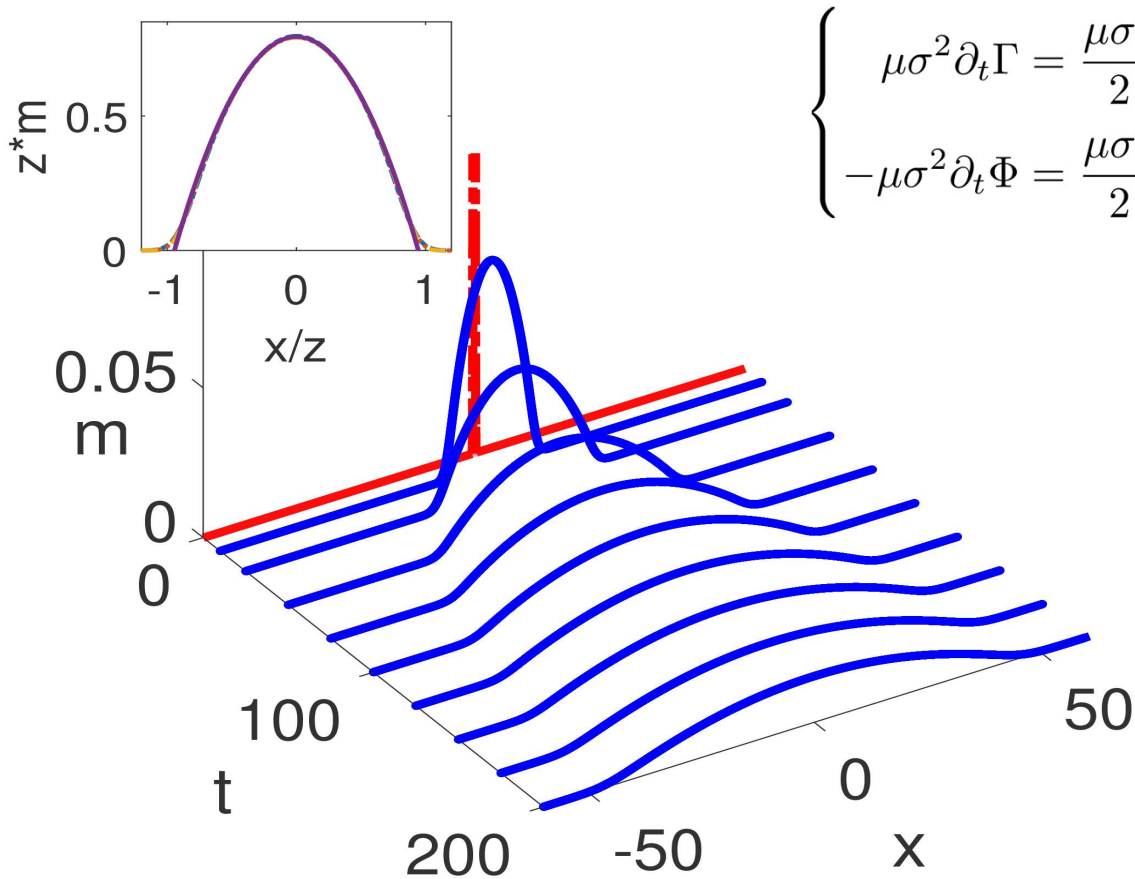


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Beginning of the game

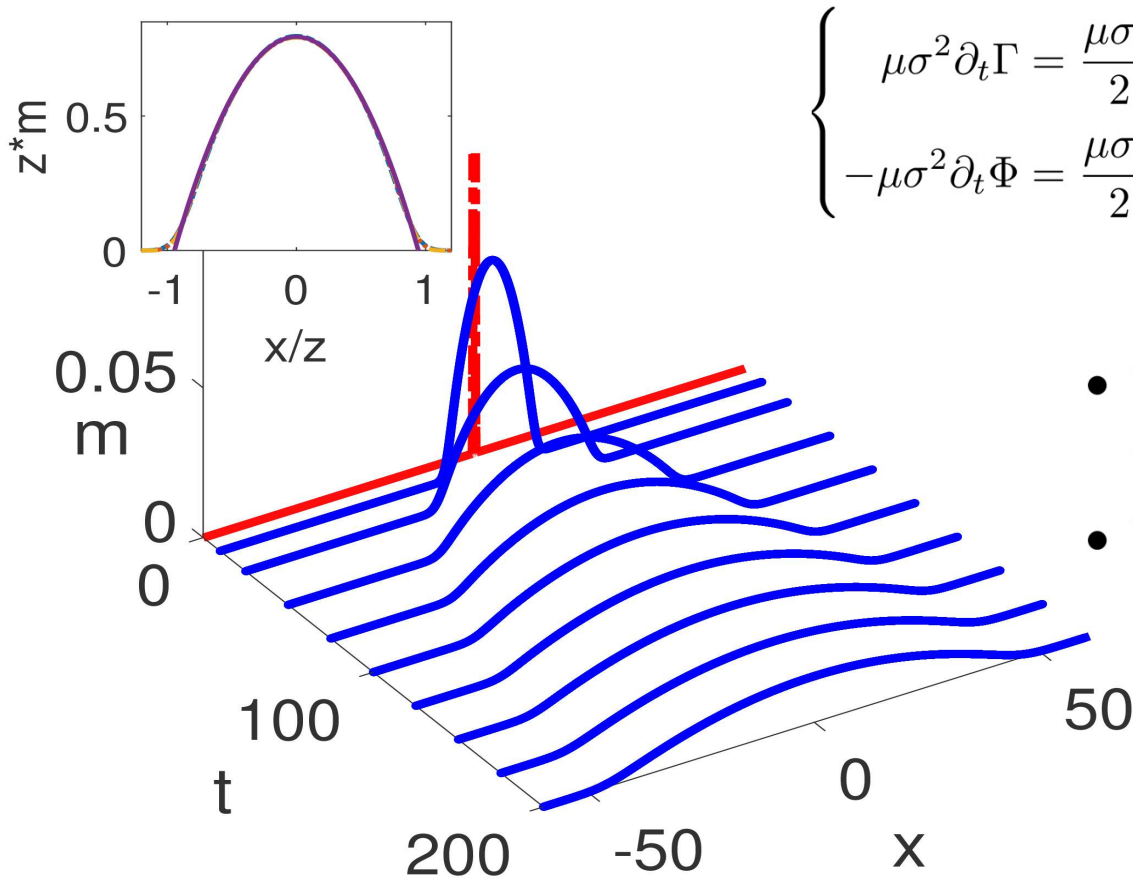


Long optimization time: numerical evidences



$$\begin{cases} \mu\sigma^2\partial_t\Gamma = \frac{\mu\sigma^4}{2}\Delta_x\Gamma + gm\Gamma + \cancel{U_0(x)\Gamma} \\ -\mu\sigma^2\partial_t\Phi = \frac{\mu\sigma^4}{2}\Delta_x\Phi + gm\Phi + \cancel{U_0(x)\Phi} \end{cases}$$

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- The density spontaneously takes a parabolic shape
- Universal scaling behaviour as $t^{2/3}$

Hydrodynamic representation

- Madelung substitution $(\Phi(x, t), \Gamma(x, t)) \longrightarrow (m(x, t), v(x, t))$:

$$v(x, t) = \frac{\sigma^2}{2} \left(\frac{\nabla \Phi(x, t)}{\Phi(x, t)} - \frac{\nabla \Gamma(x, t)}{\Gamma(x, t)} \right)$$

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- Energy

$$E = \int_{\mathbb{R}} dx \left[\frac{\mu \sigma^2}{2} \left(m \left(\frac{v}{\sigma} \right)^2 - \sigma^2 \frac{(\nabla m)^2}{4m} \right) + \frac{g}{2} m^2 + U_0 m \right]$$

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Hydrodynamic representation: parabolic ansatz

- Exact solution of hydrodynamic equations

$$\begin{cases} m = \frac{3 [z(t)^2 - x^2]}{4z(t)^3} \\ v = -\frac{z'(t)}{z(t)}x \end{cases} \quad z'(t)^2 = \frac{3|g|}{\mu} \left[\frac{1}{z(t)} + \frac{\epsilon}{\tilde{z}} \right]$$

Diapositive 44

tb25 may have to remove that one if too long
thibault bonnemain; 04/11/2022

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Diapositive 45

tb26 may have to remove that one if too long
thibault bonnemain; 04/11/2022

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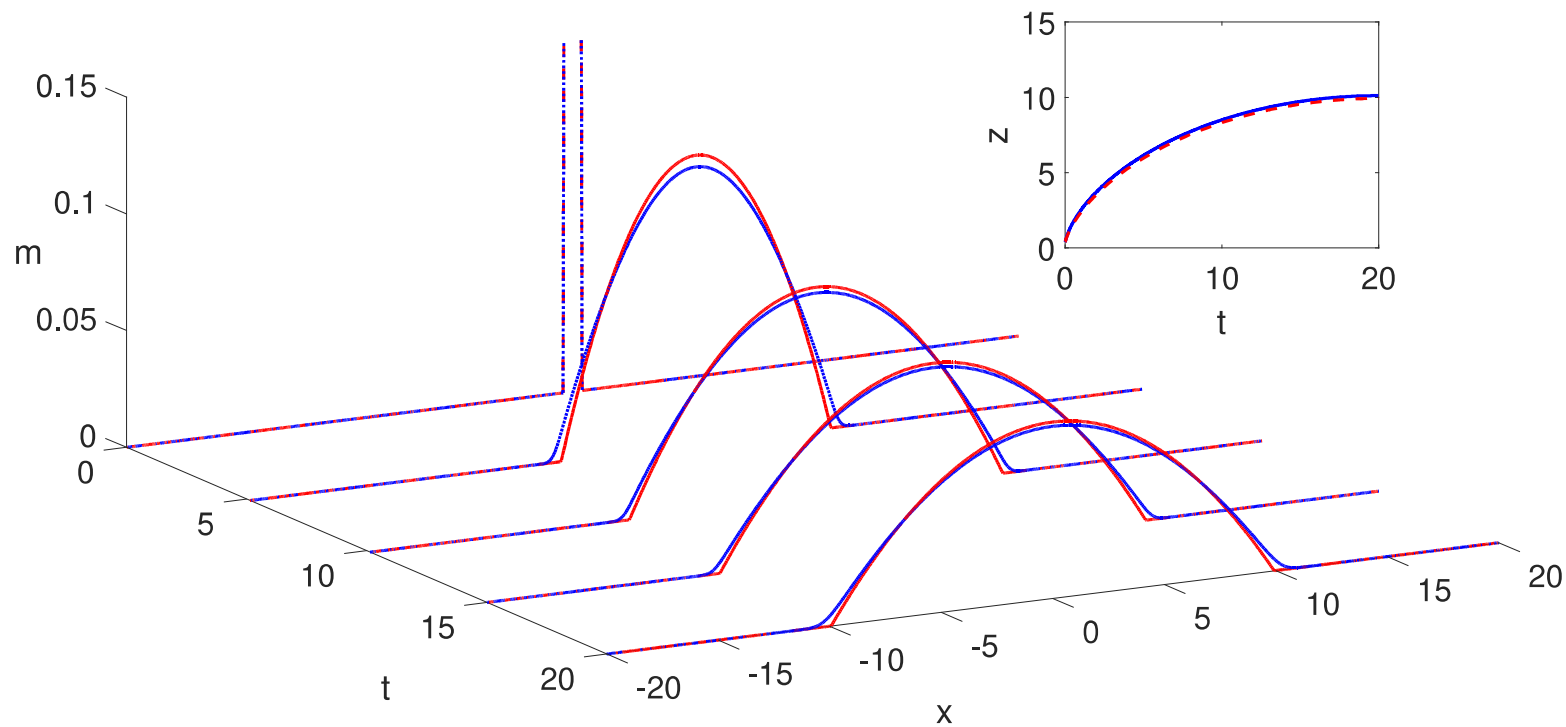
- For $\epsilon = 0$: $E = 0$ and scaling solution $t^{2/3}$
- For $\epsilon = -1$: negative energy

$$E = E^{\text{int}}(t = \tilde{T}) = \frac{3g}{10\tilde{z}} \quad \text{and} \quad \tilde{T} = \frac{\pi\tilde{z}^{3/2}}{2\sqrt{3|g|/\mu}}$$

Diapositive 46

tb27 may have to remove that one if too long
thibault bonnemain; 04/11/2022

Hydrodynamic representation: parabolic ansatz



$$\frac{\sqrt{3|g|/\mu}}{\tilde{z}^{3/2}}(t - t_0) = \arcsin \sqrt{\frac{z(t)}{\tilde{z}}} - \sqrt{\frac{z(t)}{\tilde{z}} \left(1 - \frac{z(t)}{\tilde{z}}\right)} \quad \Rightarrow \quad z \sim t^{2/3}, \quad \text{if } \frac{z}{\tilde{z}} \ll 1$$

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- Hodograph transform: linearise the system

$$\Rightarrow \text{invert} \quad (\eta(x, t), \xi(x, t)) \rightarrow (x(\eta, \xi), t(\eta, \xi))$$

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$$\vec{E} = -\nabla V, \quad \begin{cases} E_{\eta} = -\partial_{\eta} \chi = -\eta t \\ E_{\xi} = -\partial_{\xi} \chi = -2(x - \xi t) \end{cases}$$

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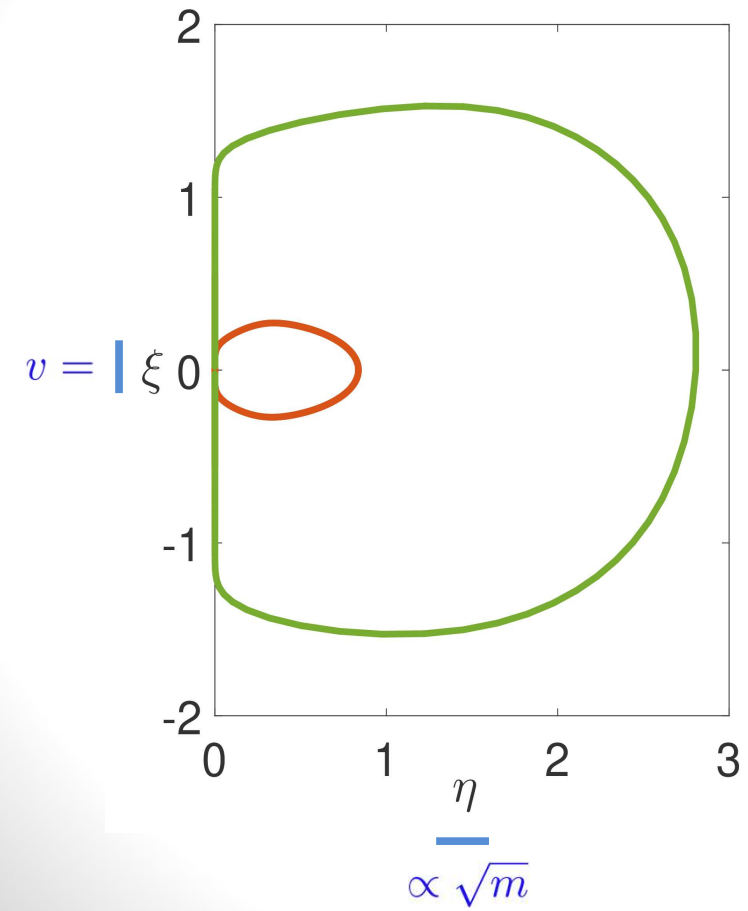
Linear equation we know how to solve!

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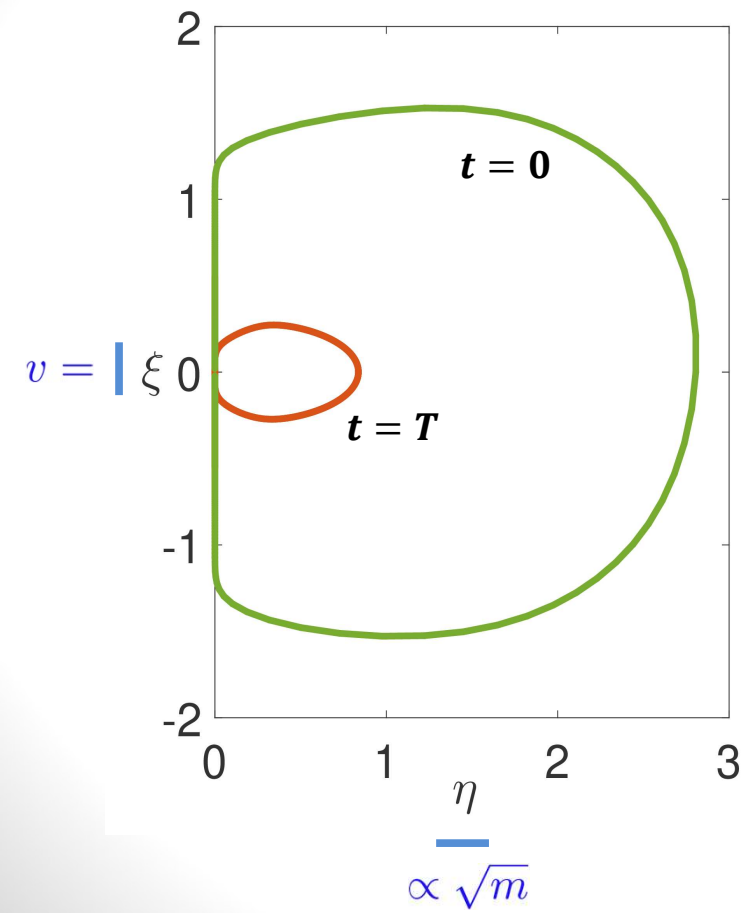
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- Price to pay: implementing boundary conditions is not so straightforward

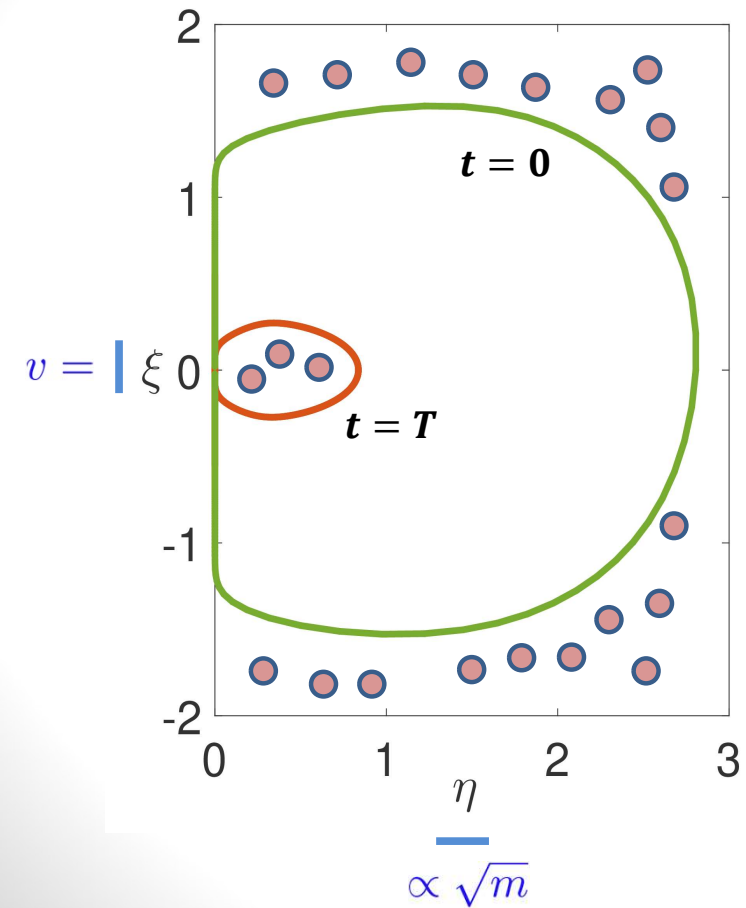
Multipole expansion



Multipole expansion

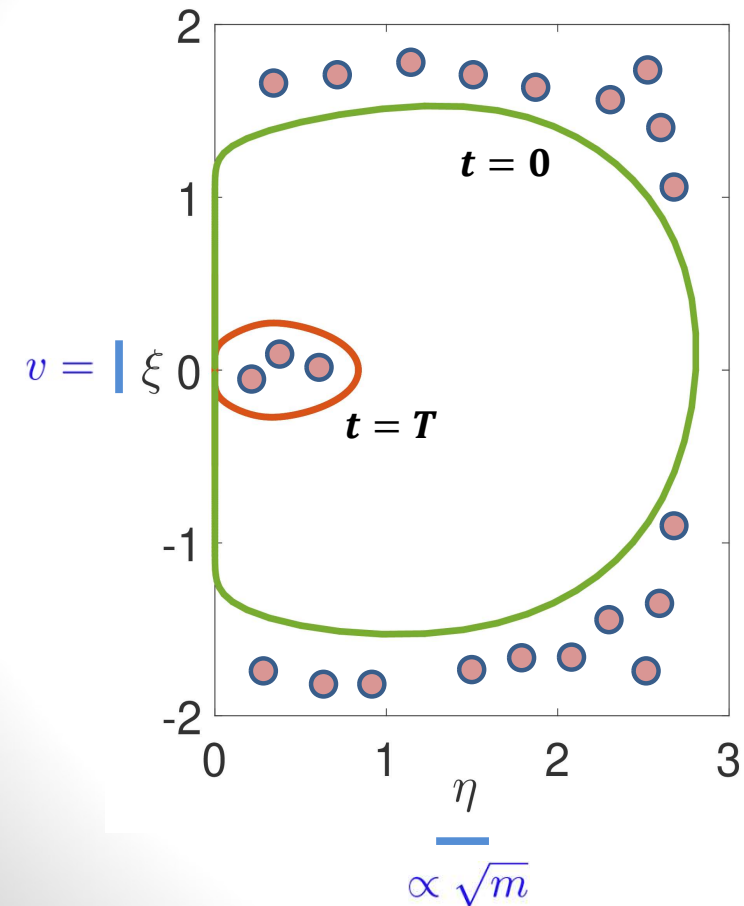


Multipole expansion



- View χ as originating from a distribution of charges located either near the origin or at infinity

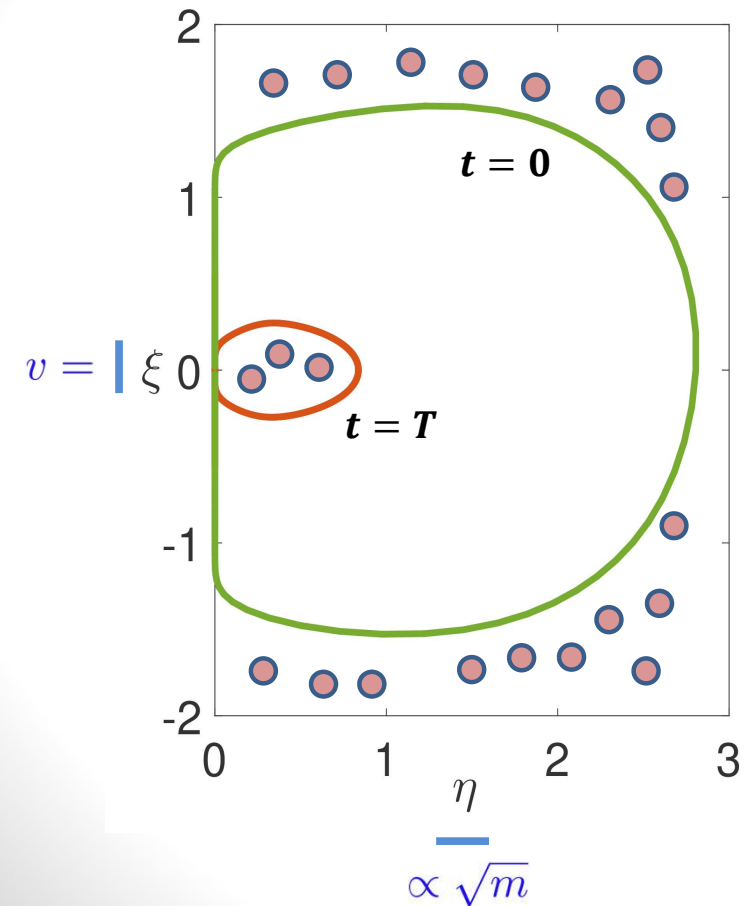
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- For $0 \ll t \ll T$: keep only the monopole

$$\chi(\eta, \xi) \approx \frac{Q_0}{r}$$

Long optimization time limit

- Monopole potential and field

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- Computing Q_0 using Gauss's law

$$Q_0 = \frac{1}{4\pi} \int_{S_{\vec{t}}} (\vec{E} \cdot \vec{n}) dS = \frac{2g}{\mu} \int_{-\infty}^{+\infty} m(x) dx \quad \underline{\quad \quad \quad} = 1$$

A link to integrability

- Quadratic Mean Field Games equations are integrable if $U_0(x) = 0$

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χ is a generating function for the conservation laws.

Quadratic MFG and integrability

- Dimensionless form

$$\begin{cases} -\partial_{t'}\Phi = +\partial_{x'x'}\Phi + 2\epsilon\nu m\Phi \\ +\partial_{t'}\Gamma = +\partial_{x'x'}\Gamma + 2\epsilon\nu m\Gamma \end{cases},$$

$$\begin{aligned} x' &= x/\nu & t' &= t/\tau \\ \nu &= \frac{\mu\sigma^4}{|g|} & \tau &= \frac{2\mu^2\sigma^6}{g^2} \end{aligned}$$

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- Lax connection

$$\partial_t U - \partial_x V + [U, V] = 0$$

$$U = \kappa_\epsilon \begin{pmatrix} \frac{\lambda}{2} & \Phi \\ \Gamma & -\frac{\lambda}{2} \end{pmatrix}, \quad V = \kappa_\epsilon \begin{pmatrix} \kappa_\epsilon \Phi \Gamma & -\partial_x \Phi \\ \partial_x \Gamma & -\kappa_\epsilon \Phi \Gamma \end{pmatrix} - \lambda U$$

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- Poisson bracket and infinite hierarchy of conservation laws

$$\{F, G\} = \int_{\mathbb{R}} \left(\frac{\delta F}{\delta \Gamma} \frac{\delta G}{\delta \Phi} - \frac{\delta F}{\delta \Phi} \frac{\delta G}{\delta \Gamma} \right) dx \quad , \quad \{Q_n, Q_m\} = 0$$

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- Soliton solutions

$$\Psi(x, t) = 2b \operatorname{sech} [2b(x + 4at - x_0)] e^{\pm 2[ax + 2(a^2 - b^2)t + \phi_0]}$$

Methodology: « search party »

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Gaussian ansatz \Rightarrow Parabolic ansatz \Rightarrow Ergodic state

$\Sigma < \nu$ $z > \nu$ $z \approx z_e$

Relaxation towards the ergodic state

Evaluation of the duration τ of the transient state

$$\left. \begin{aligned} \tilde{T} &\sim \left[\tilde{E}^{\text{int}} \right]^{-3/2} \\ E_e &= \frac{3}{2} E_e^{\text{int}} \end{aligned} \right\} \Rightarrow \tau \approx \tilde{T} (3/2)^{3/2}$$

