Generalised hydrodynamics of the KdV soliton gas.

Seminar of the LPTHE
Sorbonne Université

Thibault Bonnemain, 18th April 2024

[Based of joint work with B. Doyon and G. El]

Hydrodynamics in general

- Hydrodynamics is everywhere in physics:
  - Fluid dynamics (simple fluids Euler 1757)

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- Derived from an underlying microscopic model:

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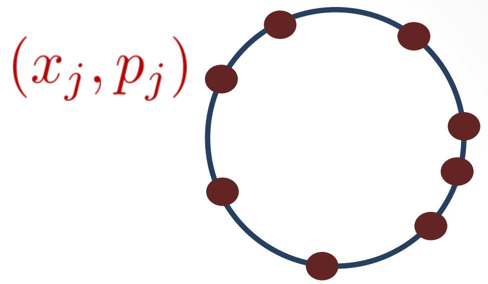
- Main ingredients:

$\Rightarrow$ local conservation laws $+$ propagation of local "equilibrium"

Conventional hydrodynamics: 1D fluid

- N particles on a circle of perimeter L



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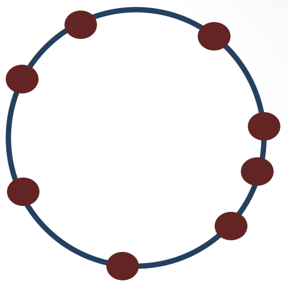

- Conservation laws

$$N$$

Number of particle

$$P = \sum_{j=1}^{N} p_j$$

Total momentum

$$E = \sum_{j=1}^{N} \frac{p_j^2}{2} + \sum_{i \neq j} V(x_i - x_j)$$

Total energy

Short range

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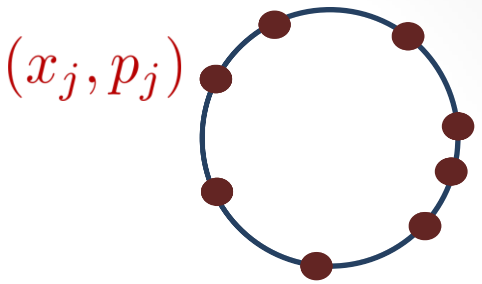

- Conservation laws

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- Local densities

$$q_0(x) = \sum_{j=1}^{N} \delta(x - x_j)$$

$$q_1(x) = \sum_{j=1}^{N} \delta(x - x_j) p_j$$

$$q_2(x) = \sum_{j=1}^{N} \delta(x - x_j) \left[\frac{p_j^2}{2} + \sum_{i \neq j} V(x_i - x_j) \right]$$

so that

$$N = \int_0^L \mathrm{d}x \; q_0(x)$$

$$P = \int_0^L \mathrm{d}x \; q_1(x)$$

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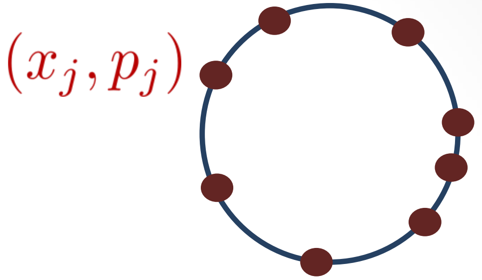

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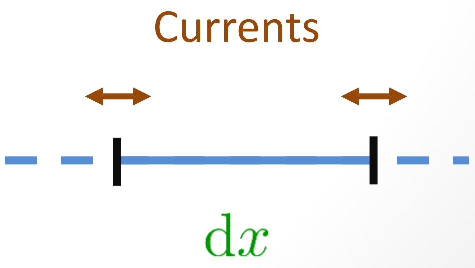
- Local conservation laws

$$\partial_t q_m(x,t) + \partial_x j_m(x,t) = 0 \; , \quad m = 0, \; 1, \; 2.$$



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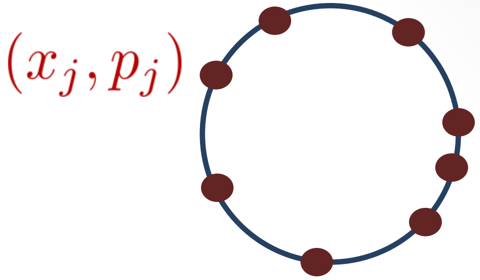

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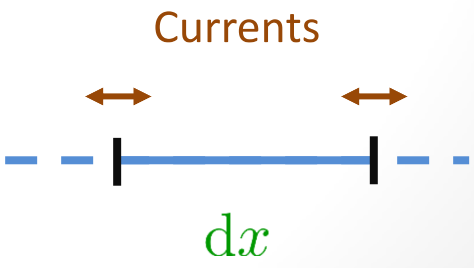

Functions on phase space or field operators

Local equilibrium

- Boltzmann 1868: micro-canonical ensemble in long time limit

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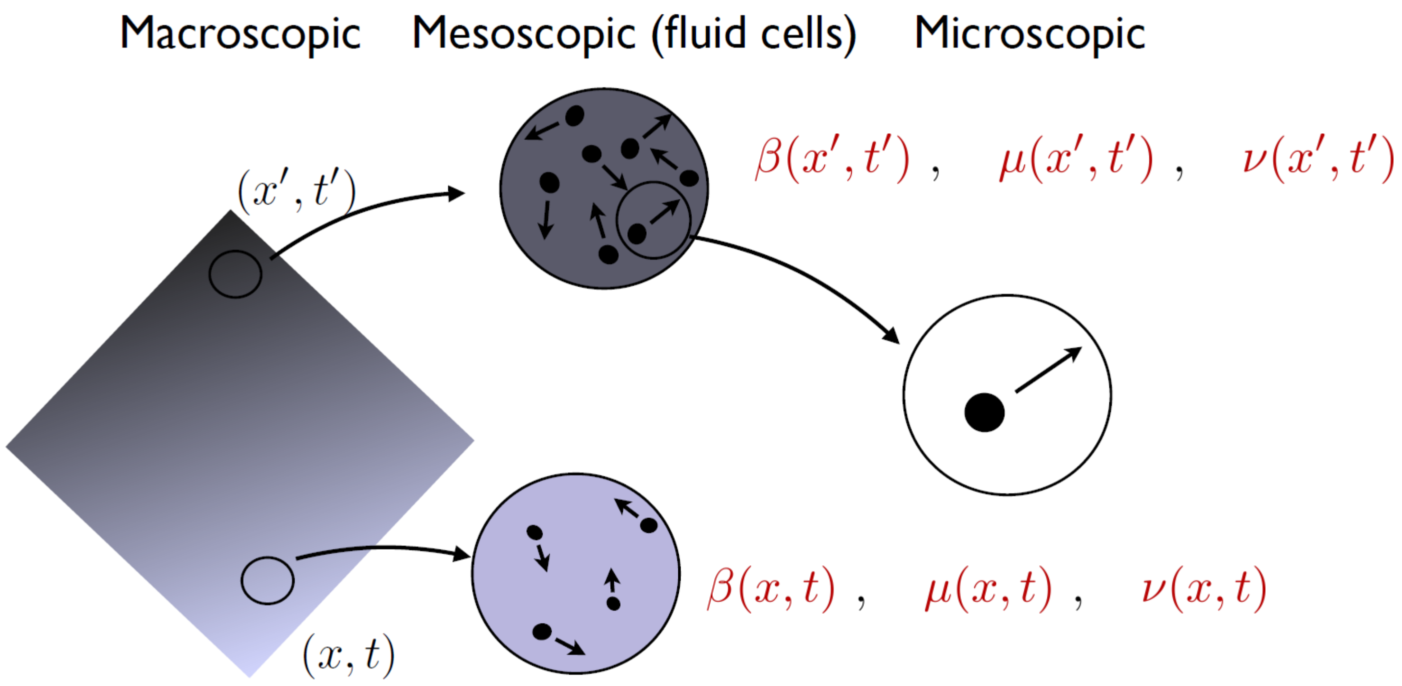

[Doyon: Lecture Notes (2020)]

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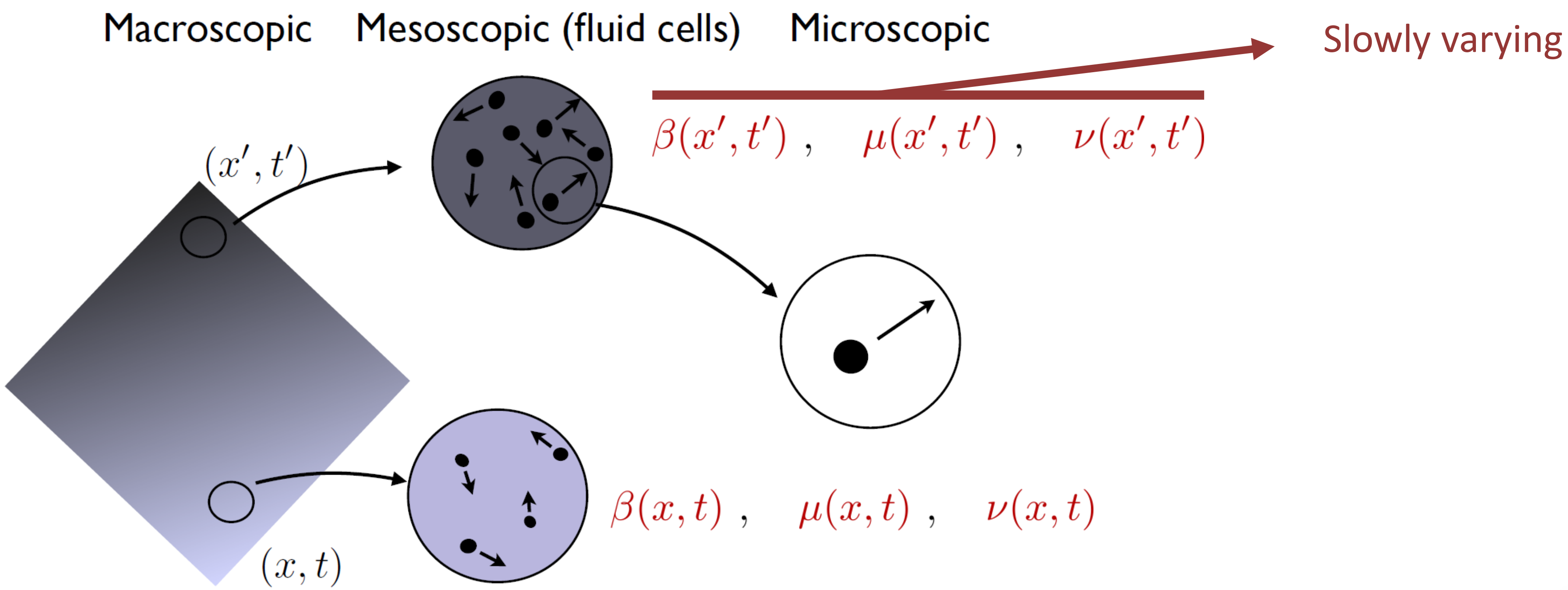

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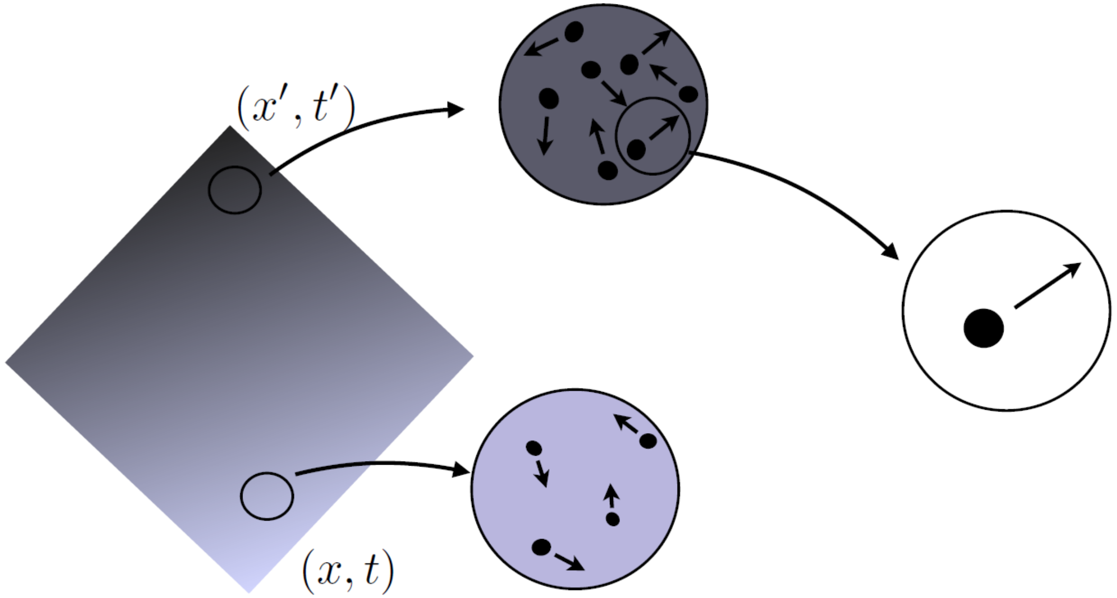
Macroscopic Mesoscopic (fluid cells) Microscopic



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$\Rightarrow$ Local GE averages

$$\bar{q}_m(x,t) \quad \text{and} \quad \bar{j}_m(x,t) \quad m = 0,\ 1,\ 2.$$

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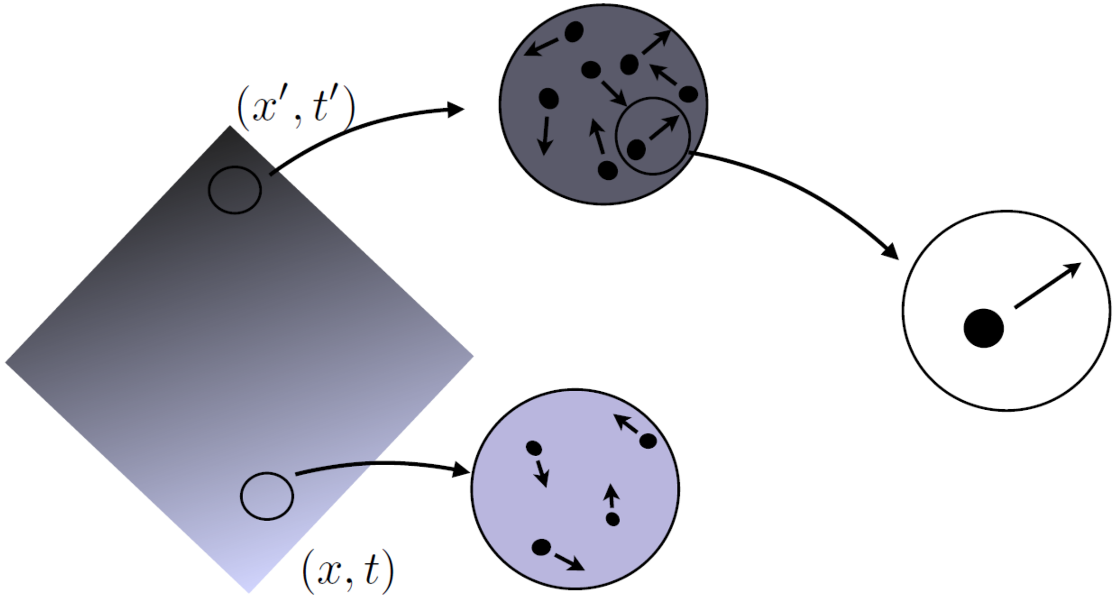

$\Rightarrow$ Local GE averages

$\bar{q}_m(x,t)$ and $\bar{j}_m(x,t)$ $\quad m = 0,\ 1,\ 2.$

$\Rightarrow$ (approx) Meso conservation law

$$\partial_t \bar{q}_m(x,t) + \partial_x \bar{j}_m(x,t) = 0$$

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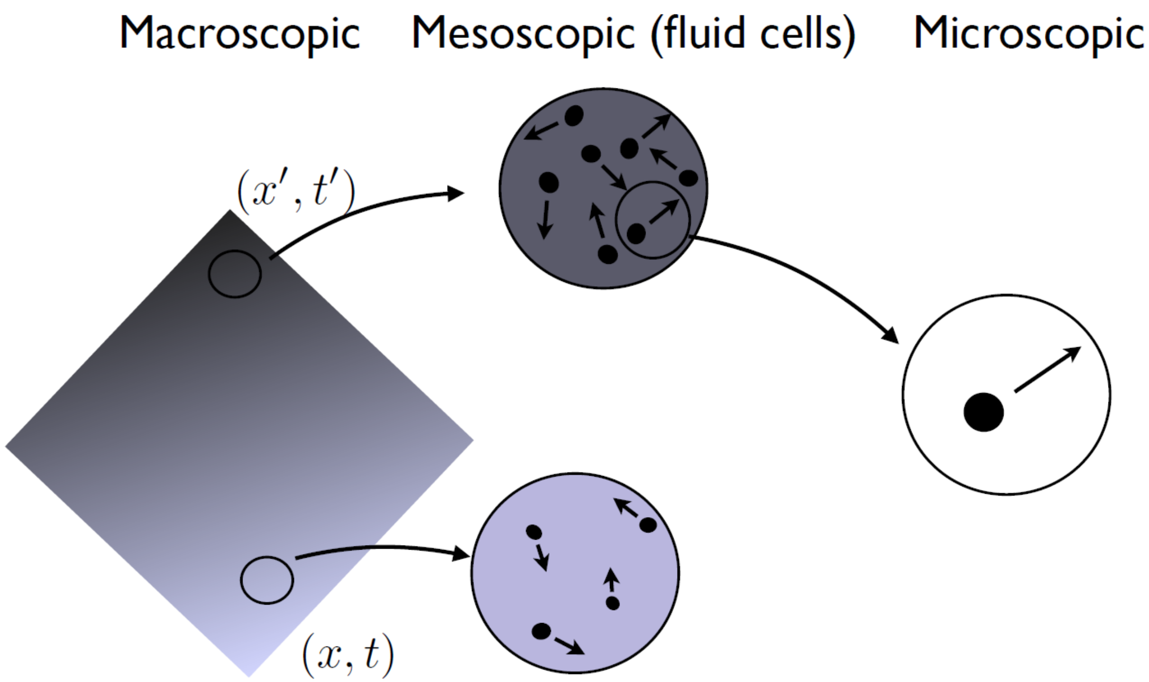

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Functions of $\{\bar{q}_n\}$'s !

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$\triangleq$ Generalised Gibbs ensembles (GGE): $\rho \propto e^{-\sum_{n=0}^{\infty} \beta_n Q_n}$

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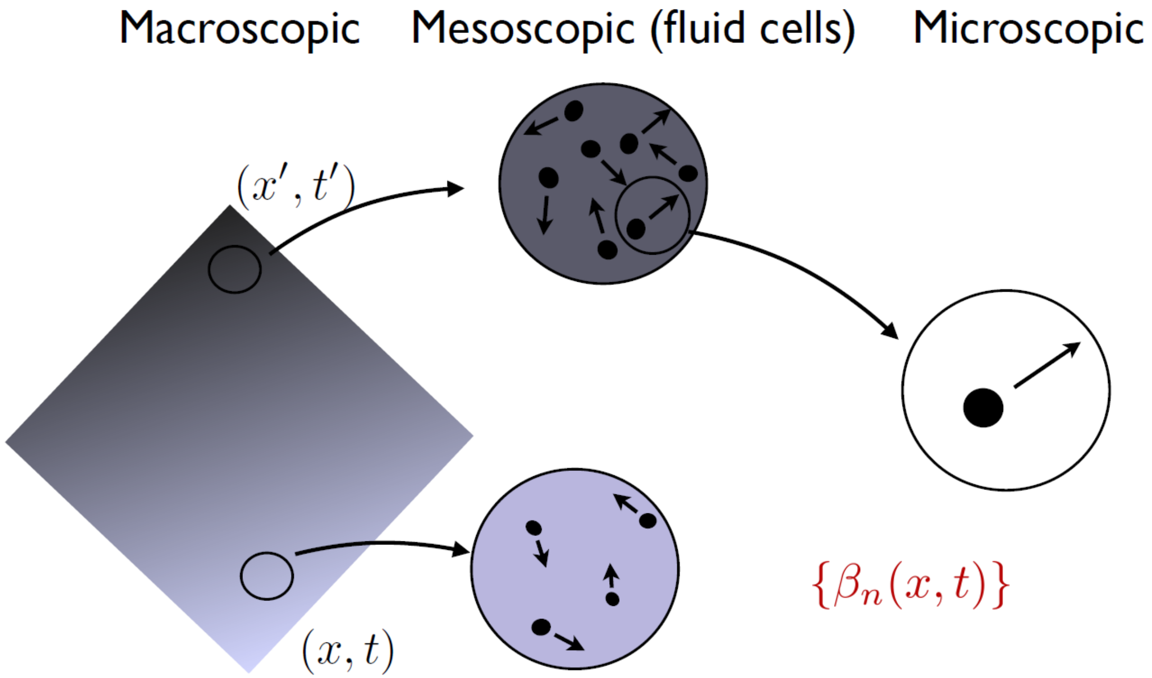

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The Korteweg-de Vries equation

- KdV: integrable, nonlinear, dispersive PDE

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- Infinite set of conservation laws

Time conserved "charges"

$$Q_n = \int \mathrm{d}x \ q_n(x,t) \ , \quad \text{and} \quad J_n = \int \mathrm{d}t \ j_n(x,t) \ ,$$

Space conserved "currents"

$$\boxed{\partial_t q_n + \partial_x j_n = 0 \ .}$$

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- Exactly solvable via Inverse Scattering Transform (IST).

[Gardner, Greene, Kruskal, Miura (1967)]

Some properties of N-soliton solutions

- Long time asymptotics of $N-$soliton solutions

$$u_N(x,t) \approx \sum_{i=1}^{N} 2\eta_i^2 \operatorname{sech}^2\left[\eta_i\left(x - 4\eta_i^2 t - x_i^{\pm}\right)\right] \quad \text{as} \quad t \to \pm\infty.$$

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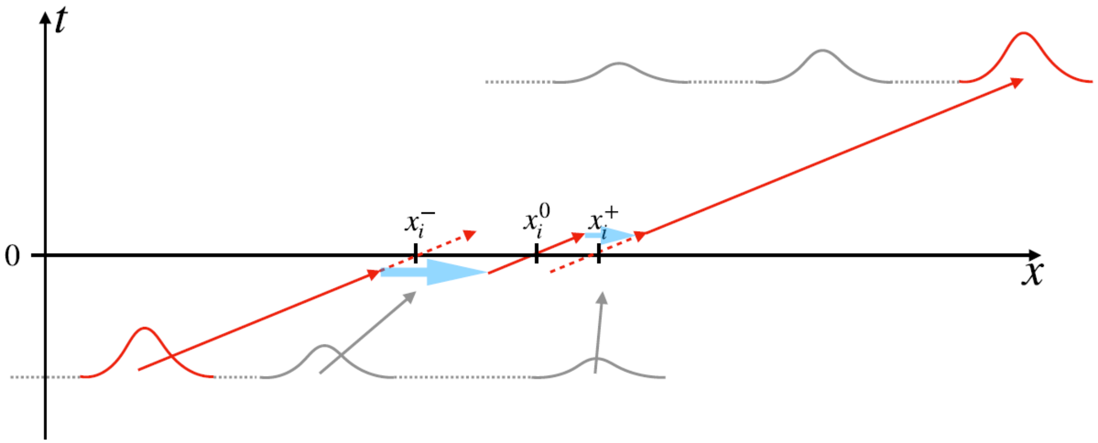
Action coordinate: η_i

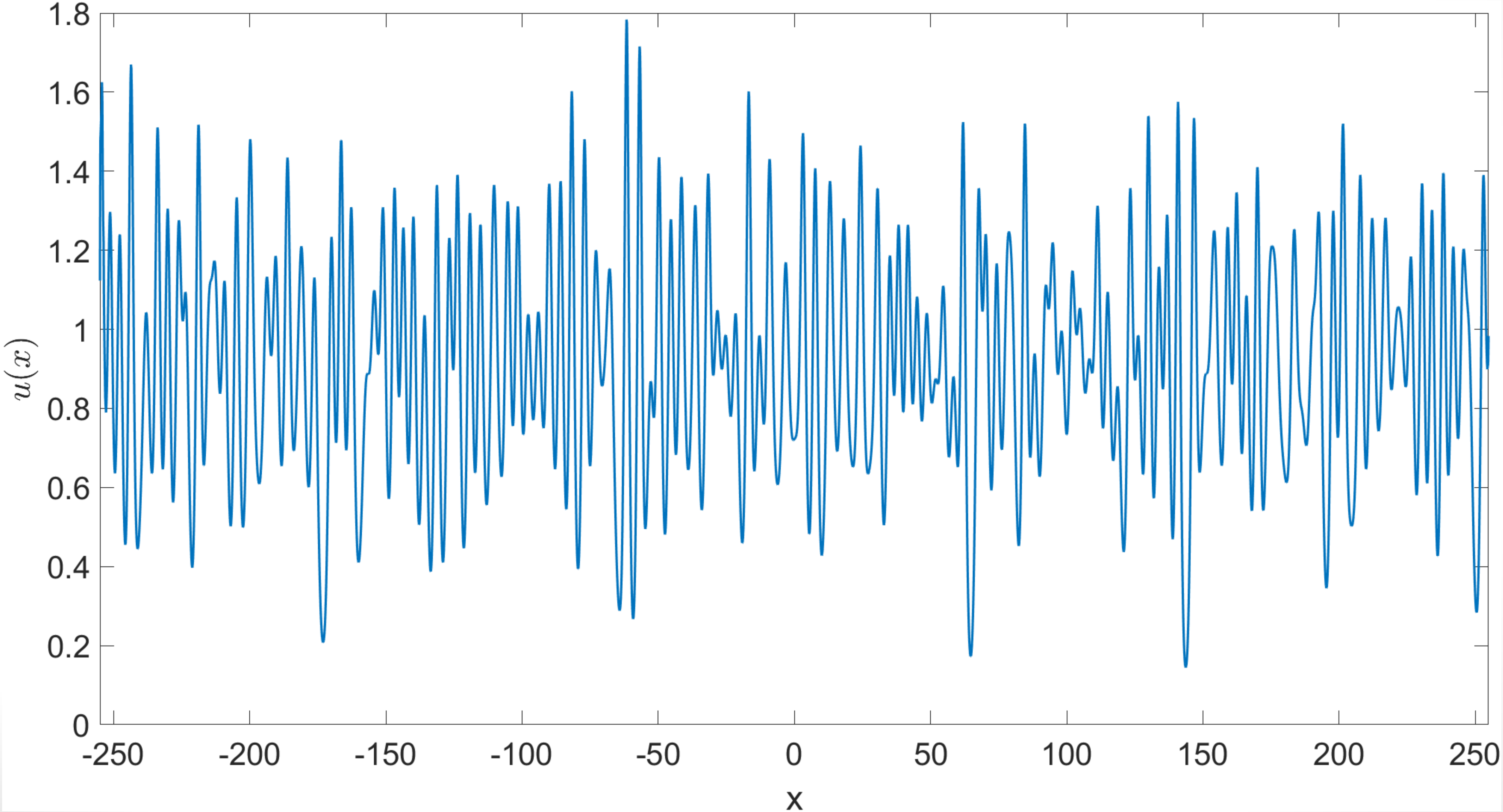
Angle coordinate: $x - 4\eta_i^2 t - x_i^{\pm}$

- Relation between asymptotic states given by scattering shift



$$x_i^+ - x_i^- = \sum_j \frac{\operatorname{sgn}(\eta_\mathrm{i} - \eta_\mathrm{j})}{\eta_i} \ln\left|\frac{\eta_i + \eta_j}{\eta_i - \eta_j}\right| .$$

Soliton gas: basic idea and motivations



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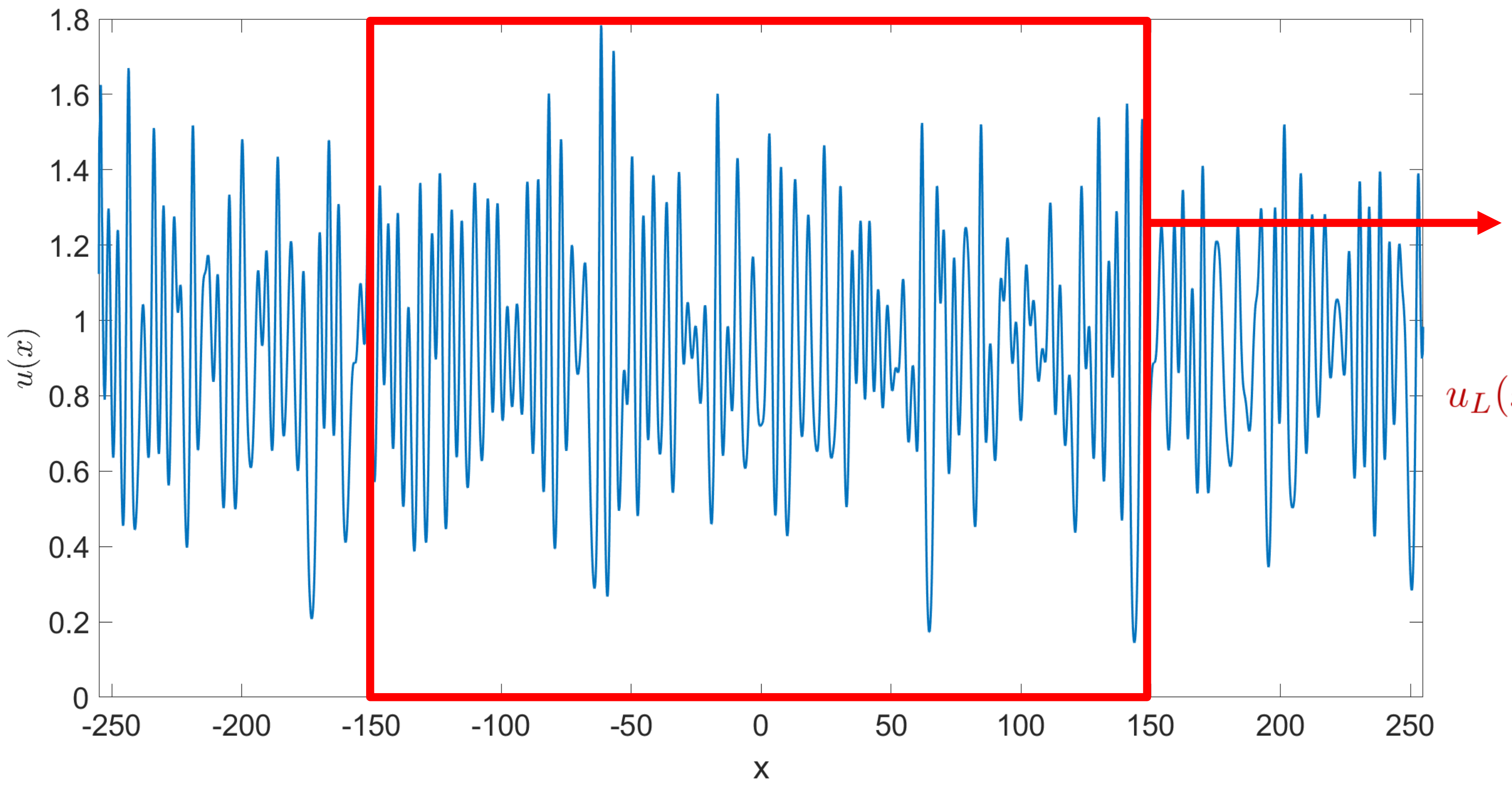

Fluid cell of size L containing N solitons

$$u_L(x,0) = \begin{cases} u(x)\ , & |x| < \dfrac{L}{2} \\ 0\ , & \text{otherwise} \end{cases}$$

Asymptotically: $u_L(x,t) \approx \sum_{i=1}^{N} 2\eta_i^2 \operatorname{sech}^2\left[\eta_i\left(x - 4\eta_i^2 t - x_i^{\pm}\right)\right]$ as $t \to \pm\infty$.

Thermodynamics

- $N-$soliton partition function can be formally written as

$$\mathcal{Z}_L = \int \mathcal{D}[u_N] \exp\left(\underbrace{S[u_N]}_{\text{Entropy}} - \underbrace{W[u_N]}_{\text{Generalised Gibbs weight}}\right) .$$

Entropy

Generalised Gibbs weight

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- $N-$soliton in asymptotic coordinates

$$\mathcal{Z}_L = \sum_{N=0}^{\infty} \frac{1}{N!} \int_{\Gamma^N \times \mathbb{R}^N} \prod_{i=1}^{N} \frac{\mathrm{d}p(\eta_i)}{2\pi} \mathrm{d}x_i^- \exp\left[-\sum_{i=1}^{N} w(\eta_i)\right] \chi\left(u_N(x, t=0) < \epsilon_x, x \notin [0, L]\right) .$$

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$$p(\eta) = 4\eta^2$$

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Constraint / Entropy

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- Assume: $\forall\ i = 1 \cdots N,\ X_i^0 = x_i^0 \in [0, L]$.
- Let i be the leftmost soliton $(x_i^0 = 0)$

$$0 \quad = \quad x_i^{\mathrm{left}} - \frac{1}{\eta_i} \sum_{\eta_j > \eta_i} \log \left| \frac{\eta_i + \eta_j}{\eta_i - \eta_j} \right| \ .$$

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Asymptotic position x_i^-

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Position at t=0

Shifts from faster solitons

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$$L \;=\; x_i^{\text{right}} + \frac{1}{\eta_i} \sum_{\eta_j < \eta_i} \log \left| \frac{\eta_i + \eta_j}{\eta_i - \eta_j} \right| \;.$$

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⇒

Asymptotic space is shorter than real space

$$L_i \equiv x_i^{\text{right}} - x_i^{\text{left}} = L - \frac{1}{\eta_i} \sum_{j \neq i} \log \left| \frac{\eta_i + \eta_j}{\eta_i - \eta_j} \right| .$$

Asymptotic space density

- Let $L_N(\eta)$ interpolate L_i

$$\mathcal{K}_N(\eta) \equiv \frac{L_N(\eta)}{L} = 1 - \frac{1}{L\eta} \sum_{j=1}^{N} \log \left| \frac{\eta + \eta_j}{\eta - \eta_j} \right| \ .$$

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- Limit $N \to \infty$, $L \to \infty$, $N/L = \varkappa$

$$\mathcal{K}(\eta) = 1 - \frac{1}{\eta}\int_\Gamma \mathrm{d}\mu\ \rho(\mu) \log\left|\frac{\eta+\mu}{\eta-\mu}\right| .$$

Aymptotic space density

Density Of States (DOS)

Asymptotic space density

- Let $L_N(\eta)$ interpolate L_i

$$\mathcal{K}_N(\eta) \equiv \frac{L_N(\eta)}{L} = 1 - \frac{1}{L\eta}\sum_{j=1}^{N} \log\left|\frac{\eta+\eta_j}{\eta-\eta_j}\right| .$$

- Limit $N \to \infty$, $L \to \infty$, $N/L = \varkappa$

$$\mathcal{K}(\eta) = 1 - \frac{1}{\eta}\int_\Gamma \mathrm{d}\mu\ \rho(\mu)\log\left|\frac{\eta+\mu}{\eta-\mu}\right| .$$

Aymptotic space density

Density Of States (DOS)

$$\frac{\varkappa}{N}\sum_{i=1}^{N}\delta(\eta-\eta_i)$$

$$\langle q_n\rangle = \int_\Gamma \mathrm{d}\eta\ \rho(\eta) h_n(\eta)$$

$h_n(\eta) = Q_n$ for a single soliton η

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Asymptotic constraint

- $N-$soliton partition function in asymptotic coordinates

$$\mathcal{Z}_L = \sum_{N=0}^{\infty} \frac{1}{N!} \int_{\Gamma^N \times \mathbb{R}^N} \prod_{i=1}^{N} \frac{\mathrm{d}p(\eta_i)}{2\pi} \mathrm{d}x_i^- \exp\left[-\sum_{i=1}^{N} w(\eta_i)\right] \chi\left(u_N(x,t=0) < \epsilon_x, x \notin [0,L]\right) \ .$$

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- Putting everything in the exponential

$$\mathcal{Z}_L = \sum_{N=0}^{\infty} \int_{\Gamma^N} \prod_{i=1}^{N} \mathrm{d}\eta_i \exp\left\{-\sum_{i=0}^{N}\left[w(\eta_i) - \log\left(\frac{4\eta_i}{\pi}\right) - \log\left[\mathcal{K}_N(\eta_i)\right] - 1 + \log \varkappa\right]\right\} \; .$$

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- Putting everything in the exponential

Jacobian

Prefactor

$$\mathcal{Z}_L = \sum_{N=0}^{\infty} \int_{\Gamma^N} \prod_{i=1}^{N} \mathrm{d}\eta_i \exp\left\{-\sum_{i=0}^{N} \left[w(\eta_i) - \log\left(\frac{4\eta_i}{\pi}\right) - \log\left[\mathcal{K}_N(\eta_i)\right] - 1 + \log \varkappa \right] \right\} \ .$$

Gibbs weight

Constraint

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$$\mathcal{Z}_L = \sum_{N=0}^{\infty} \int_{\Gamma^N} \prod_{i=1}^{N} \mathrm{d}\eta_i \exp\left\{-L \int_{\Gamma} \mathrm{d}\eta\, \tilde{\rho}(\eta) \left[w(\eta) - \log\left[\frac{4\eta \mathcal{K}_N(\eta)}{\pi}\right] - 1 + \log \varkappa \right]\right\} .$$

Empirical DOS

Thermodynamic equilibrium

- Thermodynamic limit: large deviations theory *[Varadhan (1966), Touchette (2009)]*

$$\mathcal{Z}_L \asymp \exp\left(-L\mathcal{F}^{\mathrm{MF}}[\rho^*(\eta)]\right) \ ,$$

with

$$\mathcal{F}^{\mathrm{MF}}[\rho(\eta)] = \int_\Gamma \mathrm{d}\eta \ \rho(\eta) \left[w(\eta) - \log\left[\frac{4\eta\mathcal{K}(\eta)}{\pi}\right] - 1 + \log\rho(\eta)\right] \ .$$

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Configuration entropy

[Sanov (1961)]

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- Minimisation condition (Yang-Yang equation)

$$0 = \left.\frac{\delta\mathcal{F}^{\mathrm{MF}}[\rho]}{\delta\rho(\eta)}\right|_{\rho=\rho^*} \quad \Rightarrow \quad \log\frac{4\eta\mathcal{K}(\eta)}{\pi\rho(\eta)} = w(\eta) + \int_\Gamma \mathrm{d}\mu \;\; \log\left|\frac{\eta+\mu}{\eta-\mu}\right| \frac{\rho(\mu)}{\mu\mathcal{K}(\mu)} \; .$$

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- Free energy (spatial) density

$$\mathcal{F} \equiv \mathcal{F}^{\mathrm{MF}}[\rho(\eta)] = -\int_\Gamma \mathrm{d}\mu \ \frac{\rho(\mu)}{\mathcal{K}(\mu)} \ .$$

Thermodynamic equilibrium (alternative notations)

- Minimisation condition (Yang-Yang equation)

$$\epsilon(\eta) = w(\eta) - \int_\Gamma \frac{\mathrm{d}p(\mu)}{2\pi\mu} \log\left|\frac{\eta+\mu}{\eta-\mu}\right| F(\mu) \,.$$

"pseudo-energy"

$$\epsilon = \log \frac{4\eta\mathcal{K}(\eta)}{\pi\rho(\eta)}$$

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$$F = -e^{-\epsilon(\eta)}$$

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Maxwell-Boltzmann

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Density of solitons in the asymptotic space

Maxwell-Boltzmann

Thermodynamic quantities

- Entropy density of the soliton gas

$$\mathcal{S} = \underline{\mathcal{W}} - \mathcal{F} = \int_\Gamma \mathrm{d}\eta\ \rho(\eta)\left[1 - \log n(\eta)\right] \ .$$

$$\int_\Gamma \mathrm{d}\eta\ \rho(\eta) w(\eta)$$

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- Static covariance matrix

$$\mathsf{C}_{ab} \equiv \int_\mathbb{R} \mathrm{d}x \left(\langle q_a(x) q_b(0) \rangle - \langle q_a(x) \rangle \langle q_b(0) \rangle \right) = -\frac{\partial^2 \mathcal{F}}{\partial \beta_a \partial \beta_b} \; .$$

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[Doyon (2018)]

$$h^{\mathrm{dr}}(\eta) = h(\eta) + \int_\Gamma \frac{\mathrm{d}p(\mu)}{2\pi\mu} \log \left| \frac{\eta - \mu}{\eta + \mu} \right| n(\mu) h^{\mathrm{dr}}(\mu)$$

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Statistical factor

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	MB	FD	BE	Simulations
$\mathsf{C}_{00}^{\mathrm{DC}}$	0.0235	-2.28	2.32	0.022 ± 0.003
$\mathsf{C}_{01}^{\mathrm{DC}}$	0.027	-3.18	3.23	0.024 ± 0.004
$\mathsf{C}_{11}^{\mathrm{DC}}$	0.042	-4.48	4.56	0.039 ± 0.005
$\mathsf{C}_{00}^{\mathrm{U}}$	0.22	0.028	0.41	0.2 ± 0.03
$\mathsf{C}_{01}^{\mathrm{U}}$	0.28	0.072	0.49	0.23 ± 0.04
$\mathsf{C}_{11}^{\mathrm{U}}$	0.39	0.12	0.66	0.36 ± 0.05
$\mathsf{C}_{00}^{\mathrm{L}}$	0.2	-0.05	0.45	0.2 ± 0.01
$\mathsf{C}_{01}^{\mathrm{L}}$	0.25	-0.03	0.54	0.23 ± 0.01
$\mathsf{C}_{11}^{\mathrm{L}}$	0.36	-0.03	0.75	0.34 ± 0.02

From thermodynamics to hydrodynamics

- Integrability: infinite number of conservation laws

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$$\Rightarrow \quad \boxed{\partial_t \rho(\eta;x,t) + \partial_x \left[v^{\mathrm{eff}}(\eta;x,t)\rho(\eta;x,t)\right] = 0 \ .} \quad ?$$

Alternative derivation of GHD equations

[Based on: Doyon, Spohn, Yoshimura (2017)]

- Asymptotic dynamics

$$x_j^-(t) = x_j^-(0) + 4\eta_j^2 t \ ,$$

$$\Rightarrow \quad \partial_t \rho^-(\eta; x^-, t) + 4\eta^2 \partial_{x^-} \rho^-(\eta; x^-, t) = 0 \ .$$

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- Change of metric: $\mathrm{d}x^-(\eta; x, t) = \mathcal{K}(\eta; x, t)\mathrm{d}x$

$$\partial_t n(\eta; x, t) + v^{\text{eff}}(\eta; x, t)\partial_x n(\eta; x, t) = 0 \ .$$

$$v^{\text{eff}}(\eta; x, t) = 4\eta^2 + \frac{1}{\eta}\int_\Gamma \log\left|\frac{\eta+\mu}{\eta-\mu}\right| \rho(\eta; x, t)[v^{\text{eff}}(\eta; x, t) - v^{\text{eff}}(\eta; x, t)]\mathrm{d}\mu \ .$$

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- Continuity equation for the DOS

$$\partial_t \rho(\eta; x, t) + \partial_x \left[\rho(\eta; x, t) v^{\text{eff}}(\eta; x, t)\right] = 0 \ .$$

Concluding remarks and perspectives

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- Perspective: integrability of the GHD equations (generalised hodograph method, Hamiltonian formalism...).

GHD in a nutshell

	Homogeneous - Stationary		Inhomogeneous - Dynamical
Finite number of conserved quantities	Statistical Thermodynamics	⇒	Hydrodynamics
	⇓		⇓
Infinite number of conserved quantities	Generalized Gibbs Ensembles	⇒	Generalized Hydrodynamics