Introduction to Generalized Hydrodynamics

PhLAM Seminar at the University of Lille

Thibault Bonnemain, 7th April 2023

Hydrodynamics in general

- Hydrodynamics is everywhere in physics:
  - Fluid dynamics (simple fluids Euler 1757)

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- Derived from an underlying microscopic model:

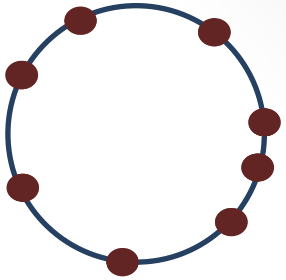
$$\Rightarrow \quad \text{field theories or many-particle systems}$$

- Main ingredients:

$$\Rightarrow \quad \text{local conservation laws} \quad + \quad \text{propagation of local “equilibrium”}$$

Conventional hydrodynamics: 1D fluid

- N particles on a circle of perimeter L



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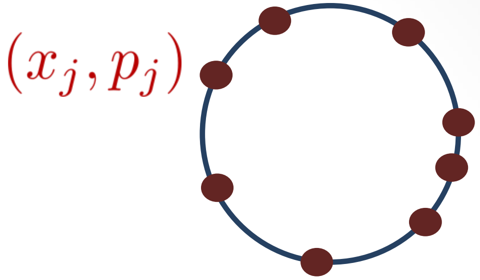


- N particles on a circle of perimeter L
- Conservation laws

$$N$$

Number of particle

$$P = \sum_{j=1}^{N} p_j$$

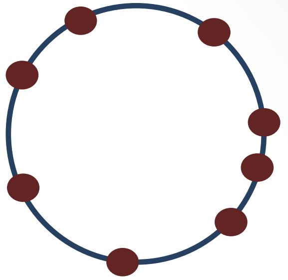
Total momentum

$$E = \sum_{j=1}^{N} \frac{p_j^2}{2} + \sum_{i \neq j} V(x_i - x_j)$$

Total energy

Short range

Conventional hydrodynamics: 1D fluid



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- Local densities

$$q_0(x) = \sum_{j=1}^{N} \delta(x - x_j)$$

$$q_1(x) = \sum_{j=1}^{N} \delta(x - x_j) p_j$$

$$q_2(x) = \sum_{j=1}^{N} \delta(x - x_j) \left[\frac{p_j^2}{2} + \sum_{i \neq j} V(x_i - x_j) \right]$$

so that

$$N = \int_0^L \mathrm{d}x \; q_0(x)$$

$$P = \int_0^L \mathrm{d}x \; q_1(x)$$

$$E = \int_0^L \mathrm{d}x \; q_2(x)$$

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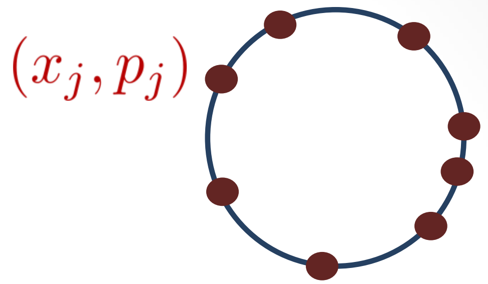

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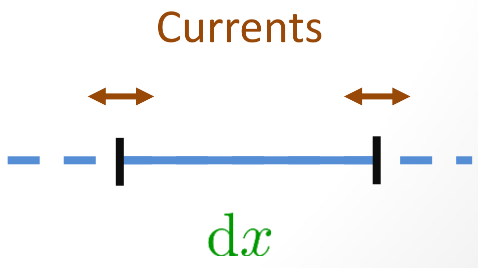
- Local conservation laws

$$\partial_t q_m(x,t) + \partial_x j_m(x,t) = 0 \, , \quad m = 0, \; 1, \; 2.$$



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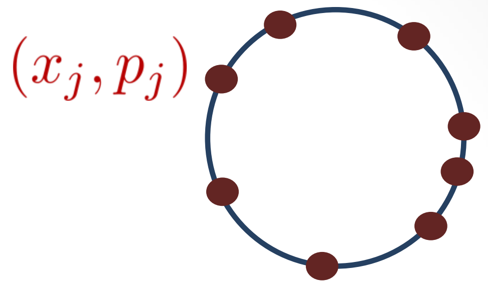

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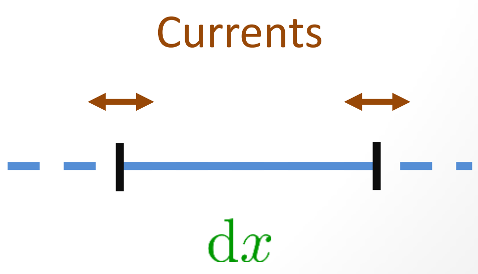

Functions on phase space or field operators

Local equilibrium

- Boltzmann 1868: micro-canonical ensemble in long time limit

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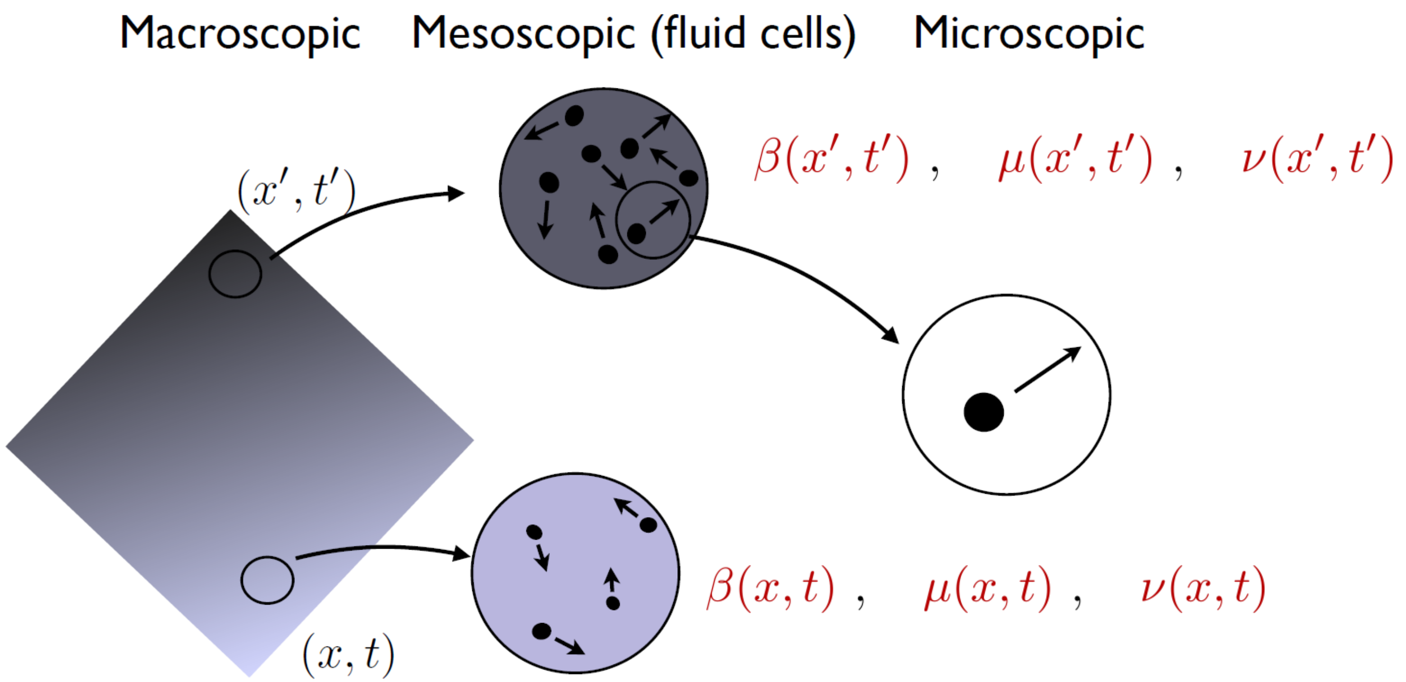
- Hydrodynamic principle: separation of scales and propagation of local GE



[Doyon: Lecture Notes (2020)]

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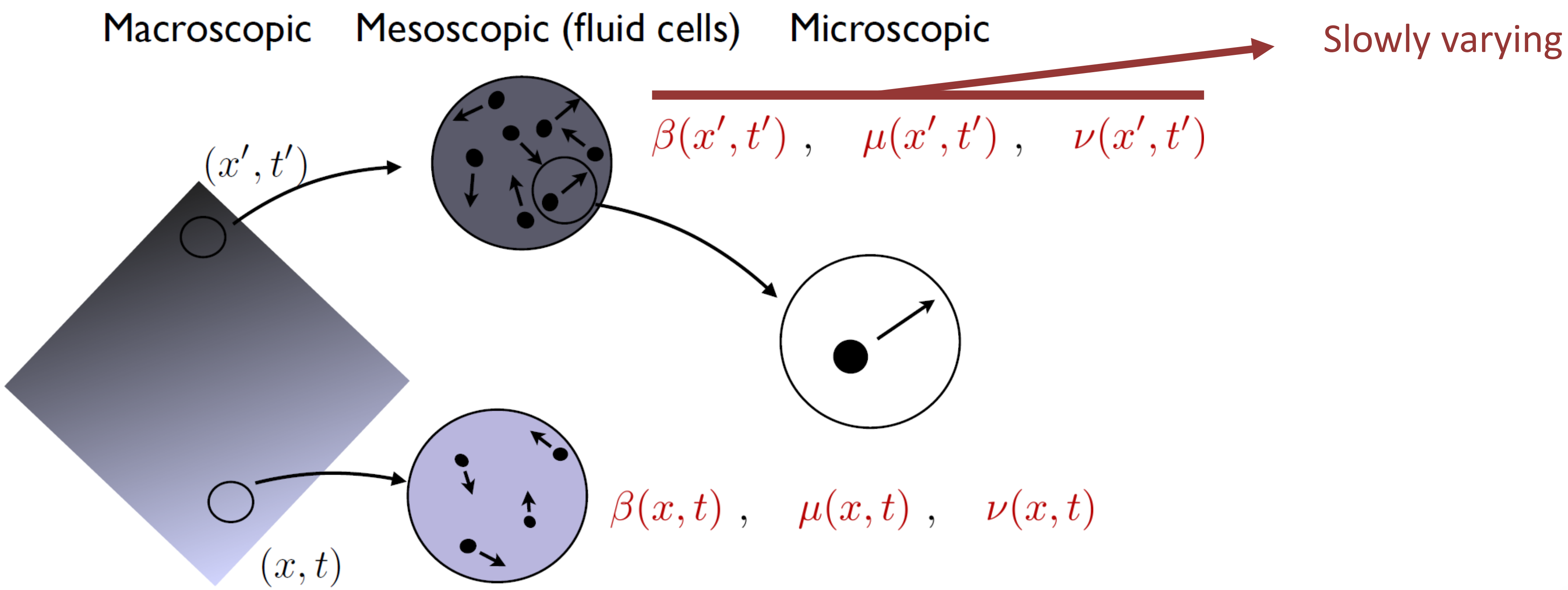

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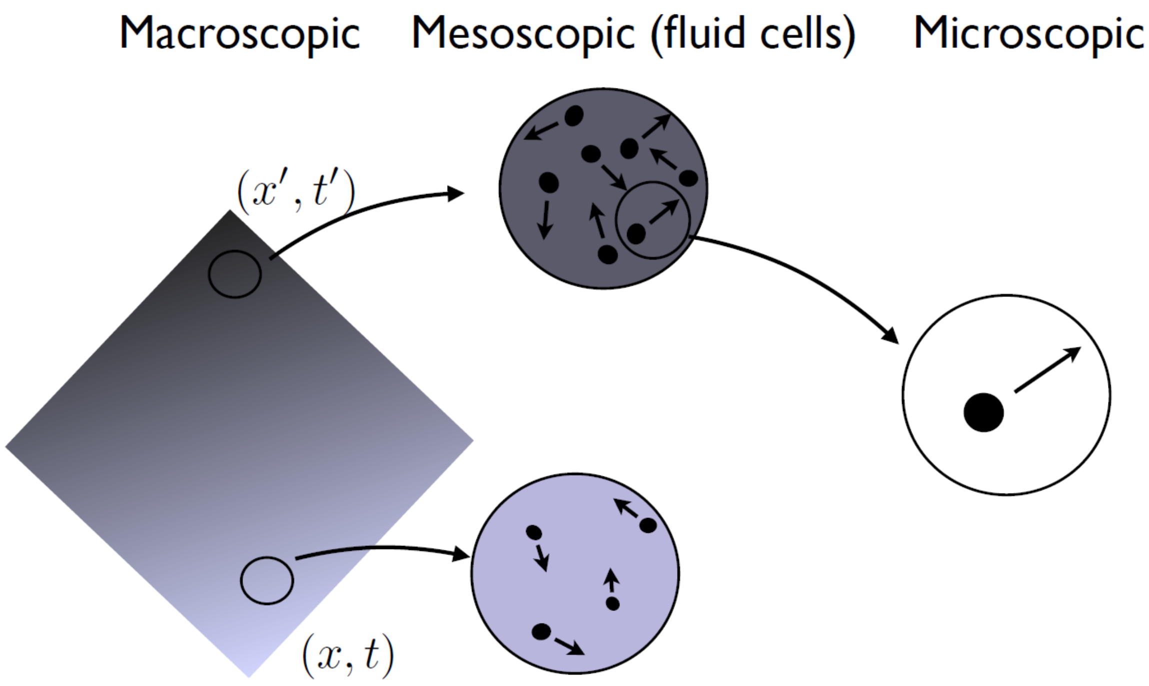

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$\Rightarrow$ Local GE averages

$$\bar{q}_m(x,t) \quad \text{and} \quad \bar{j}_m(x,t)$$

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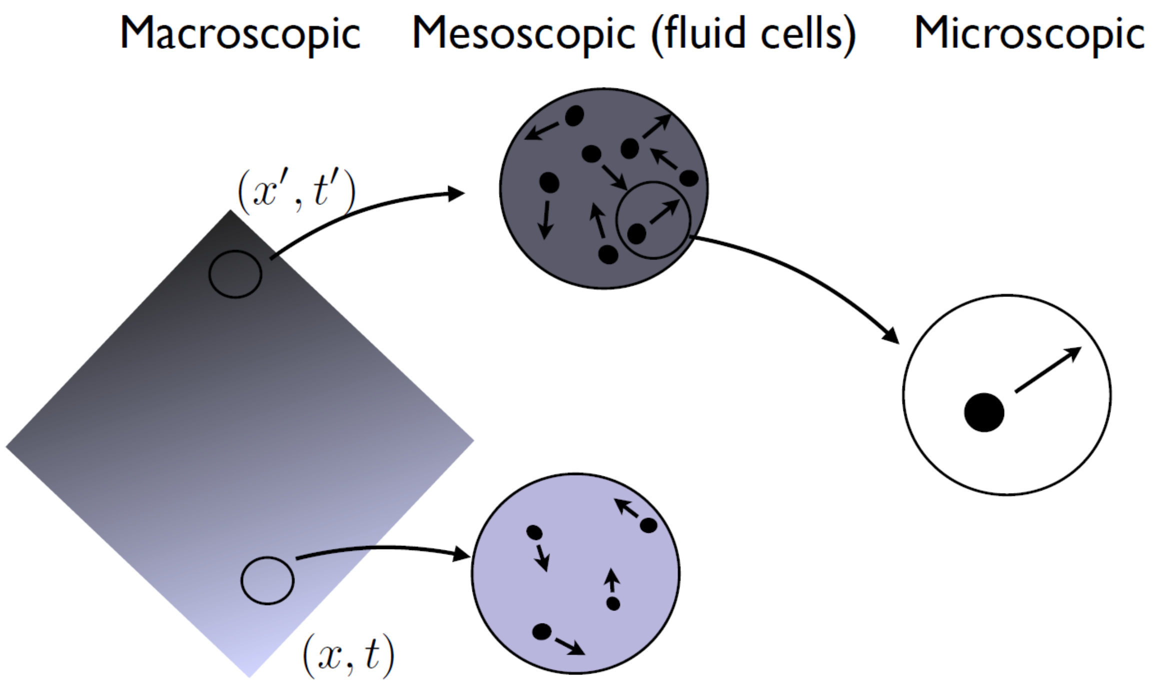

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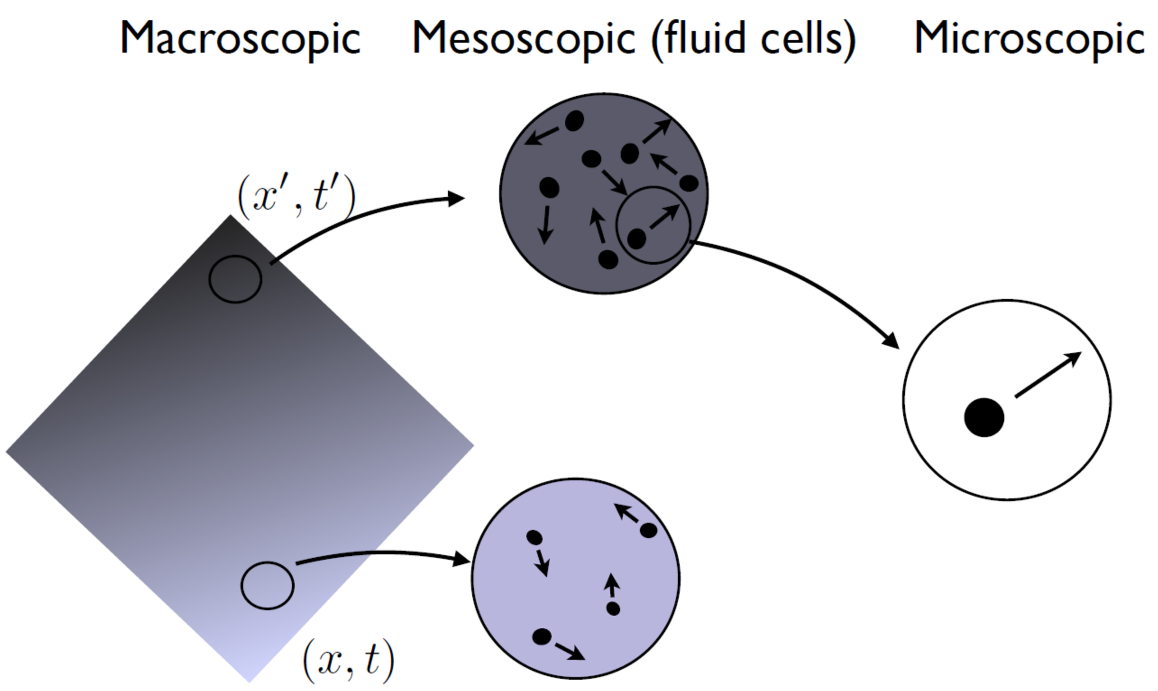

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Functions of $\{\bar{q}_n\}$'s !

More conservation laws?

- Integrable systems: infinite number of conserved quantities!

  E.g. free particles, Lieb-Liniger, Toda, Camassa-Holm, KdV, NLS …

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- GE are replaced by generalized Gibbs Ensemble (GGE)

[Mossel, Caux (2012)]

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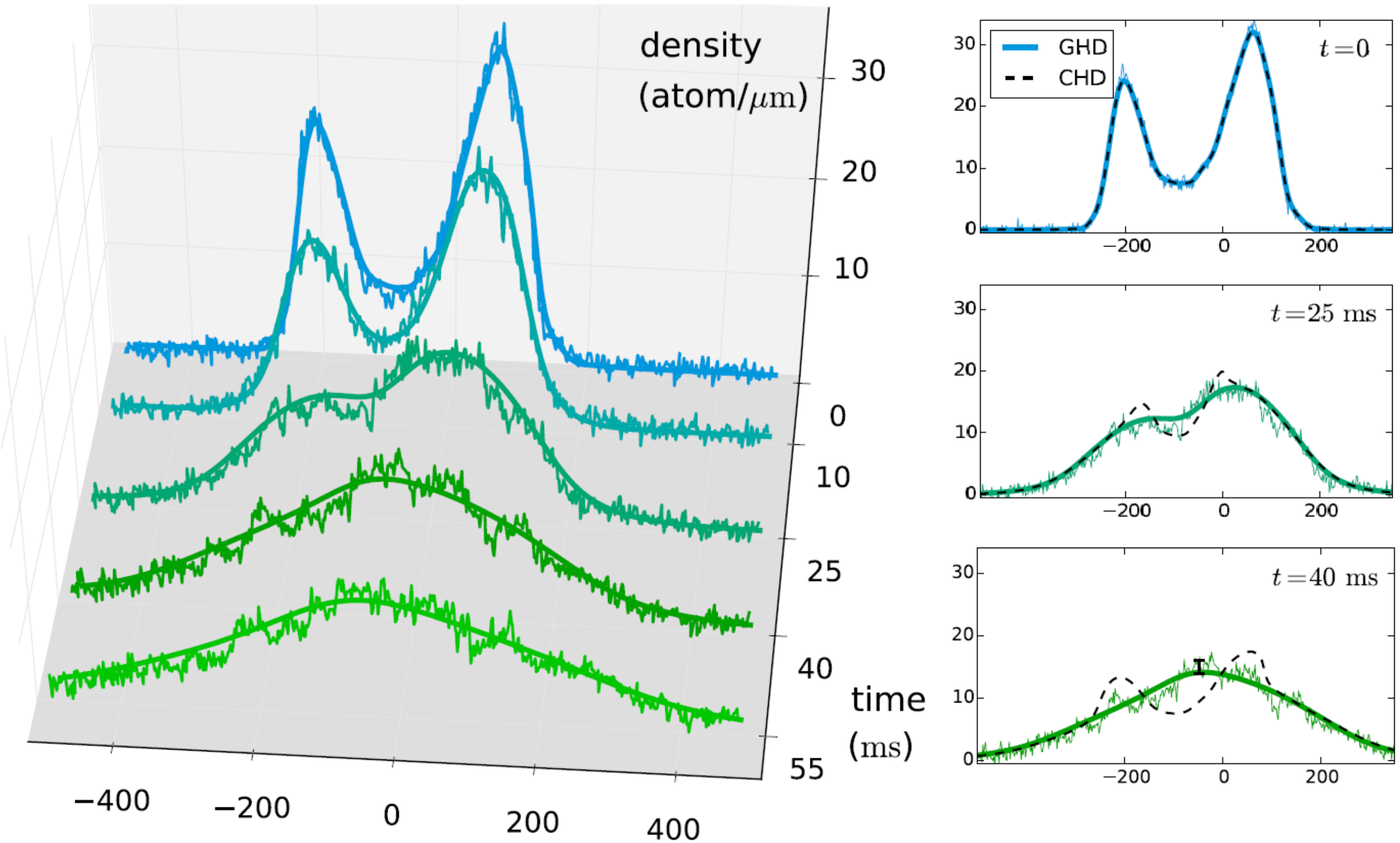

1D Bose gas is described by the Lieb-Liniger model

[Schemmer, Bouchoule, Doyon, Dubail (2019)]

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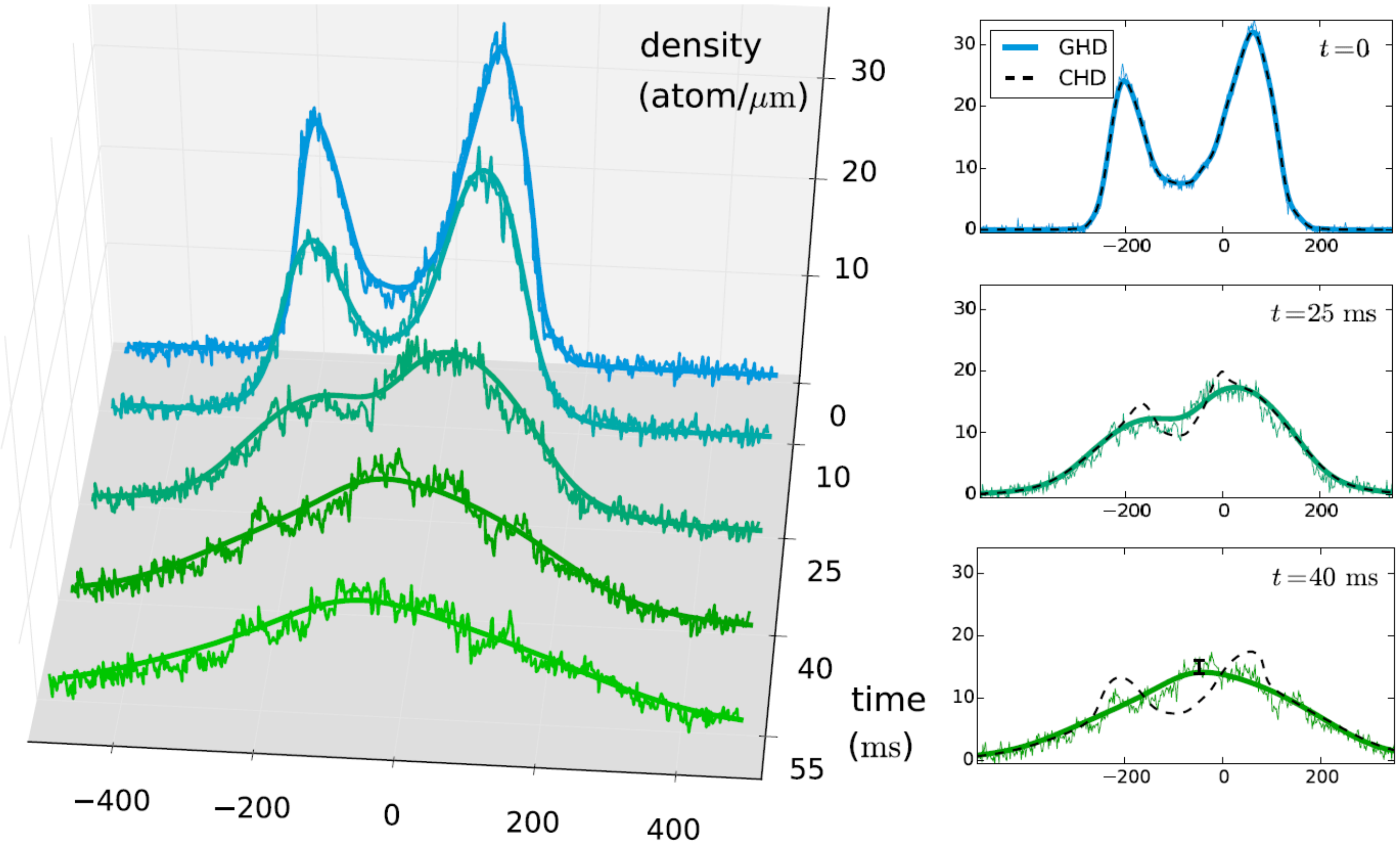

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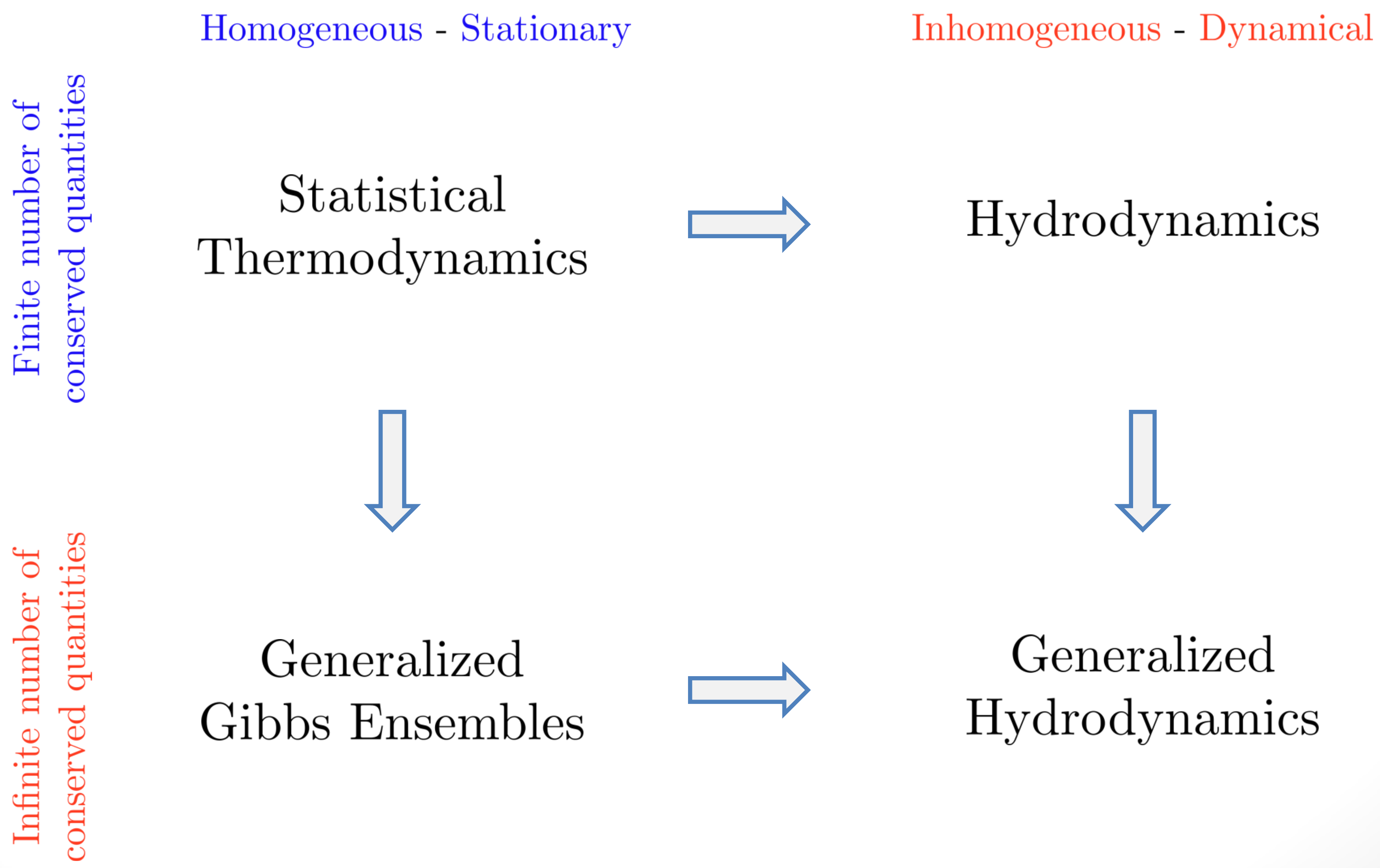
GHD experimentally observed on an atom chip

[Schemmer, Bouchoule, Doyon, Dubail (2019)]

GHD in a nutshell



A concrete example: Lieb-Liniger model

[Lieb, Liniger (1963)]

- Integrable model for a system of interacting bosons in 1D

$$H = -\frac{1}{2m}\sum_{j}\frac{\partial^2}{\partial {x_j}^2} + g\sum_{j<k}\delta(x_j - x_k)$$

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- Two particle time independent Schrödinger equation

$$\left[-\frac{1}{2}\left(\frac{\partial^2}{\partial {x_1}^2} + \frac{\partial^2}{\partial {x_2}^2}\right) + g\delta(x_1 - x_2)\right]\Psi(x_1, x_2) = E\Psi(x_1, x_2)$$

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For give momentum $P = k_1 + k_2$ and energy $E = ({k_1}^2 + {k_2}^2)/2$

$$\Psi(x_1, x_2) = A_{12}\exp(i[k_1 x_1 + k_2 x_2]) + A_{21}\exp(i[k_1 x_2 + k_2 x_1]), \quad x_1 < x_2$$

$$\frac{A_{21}}{A_{12}} = \frac{k_1 - k_2 - ig}{k_1 - k_2 + ig} \equiv \exp\left[-i\theta(k_1 - k_2)\right]$$

Bethe ansatz for N particles

[Lieb, Liniger (1963)]

- Superposition of plane waves

$$\Psi(x_1, x_2, \ldots, x_N) = \sum_P A_P \exp\left(i \sum_{j=1}^{N} k_{P(j)} x_j\right), \quad x_1 < x_2 < \cdots x_N$$

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- Amplitudes given by 2-body phase shift

$$\frac{A_{P'}}{A_P} = \frac{k_{P(j)} - k_{P(j+1)} - ig}{k_{P(j)} - k_{P(j+1)} + ig} = \exp\left[-i\theta(k_{P(j)} - k_{P(j+1)}) \right]$$

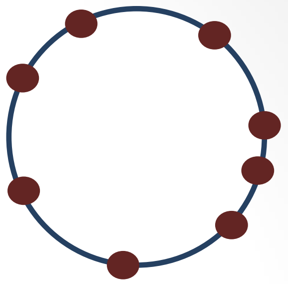
where

$$P' = (P(1), P(2), \ldots, P(j-1), P(j+1), P(j), P(j+2), \ldots, P(N))$$

Quantization conditions

- Periodic boundary conditions

$$\Psi(x_1,\ldots,x_j,\ldots,x_N)=\Psi(x_1,\ldots,x_j+L,\ldots,x_N)$$



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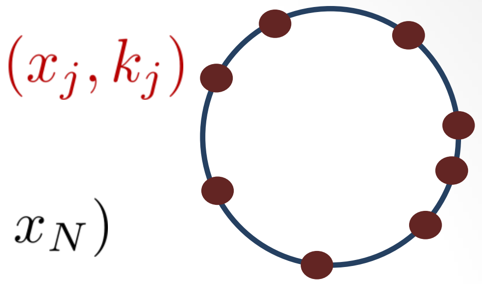

$$\Psi(x_1, \ldots, x_j, \ldots, x_N) = \Psi(x_1, \ldots, x_j + L, \ldots, x_N)$$

- Free particles

$$e^{ik_j L} = 1$$

Quantization conditions

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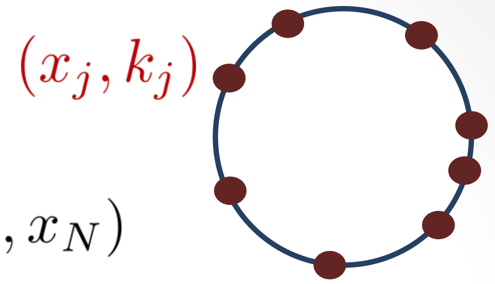

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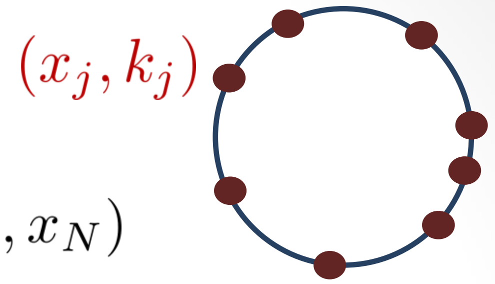

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- More practical representation

$$k_j L = 2\pi I_j + \sum_{\substack{n=1 \\ n \neq j}}^{N} \theta(k_j - k_n)$$

Thermodynamic limit

[Yang, Yang (1969)]

- Introduce the counting function

$$Lc(k) = kL - \sum_{\substack{n=1 \\ n \neq j}}^{N} \theta(k - k_n)$$

so that $Lc(k_j) = 2\pi I_j$ corresponds to a particle.

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- $E = \sum_{j=1}^{N} {k_j}^2 \quad \Rightarrow \quad$ ground state for set of smallest $\{I_j\}$

$$I_j = -(N-1)/2, -(N-3)/2, \ldots, -1, 0, 1 \ldots (N-1)/2$$

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- Excitations as particle-hole pairs

$$Lc(h_j) = 2\pi J_j \ , \quad J_j \in (\mathbb{Z} \setminus \{I_j\})$$

corresponds to a hole.

Thermodynamic limit

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- Introduce particles and holes densities

$$L\rho_p(k)\mathrm{d}k = \#\text{ of } k\text{'s in dk}$$
$$L\rho_h(k)\mathrm{d}k = \#\text{ of } h\text{'s in dk}$$

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- \# of k's and h's: \# of times Lc ranges over $2\pi I$'s and $2\pi J$'s

$$\frac{\mathrm{d}c(k)}{\mathrm{d}k} = 2\pi\left[\rho_p(k) + \rho_h(k)\right] \equiv 2\pi\rho_s(k)$$

Density of available states

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Density of available states

- Thermodynamic Bethe equation

$$Lc(k) = kL - \sum_{\substack{n=1\\ n\neq j}}^{N} \theta(k-k_n) \quad \rightarrow \quad 2\pi\rho_s(k) = 1 + \int \mathrm{d}p\, \frac{2g}{g^2+(k-p)^2}\, \rho_p(p)$$

$$\varphi(k-p) \equiv \theta'(k-p)$$

Thermodynamic quantities

- Conserved quantities

$$Q_n = \int \mathrm{d}x \, \langle q_n(x) \rangle = L \int \mathrm{d}k \; \rho_p(k) h_n(k)$$

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- Because of holes many configurations of same ρ_p, ρ_h

$$\frac{[L(\rho + \rho_h)dk]!}{[L\rho dk]![L\rho_h dk]!}$$

available states in dk

particles

holes

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- Define entropy

$$S = L \int \mathrm{d}k\ [(\rho_p + \rho_h)\log(\rho_p + \rho_h) - \rho_p \log \rho_p - \rho_h \log \rho_h]$$

Thermodynamic equilibrium

- From these elements define a partition function

$$Z = \int \mathcal{D}[\rho_p] \; \exp\left(S\left[\rho_p, \rho_h[\rho_p]\right] - \sum_{n=0}^{\infty} \beta_n Q_n[\rho_p] \right)$$

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$$\epsilon(k) = w(k) - \int \mathrm{d}p \ \varphi(k-p) F\left(\epsilon(k)\right)$$

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"pseudo-energy"

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$$\epsilon \equiv \log \frac{\rho_h}{\rho_p} \qquad\qquad w \equiv \sum_{n=0}^{\infty} \beta_n h_n$$

"pseudo-energy" generalised Gibbs weight

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$$Z = \int \mathcal{D}[\rho_p] \; \exp\left(S\left[\rho_p, \rho_h[\rho_p]\right] - \sum_{n=0}^{\infty} \beta_n Q_n[\rho_p] \right)$$

- Equilibrium given by saddle point condition: Yang-Yang equation

$$\boxed{\epsilon(k) = w(k) - \int \mathrm{d}p \; \varphi(k-p) F\left(\epsilon(k)\right)}$$

$$\epsilon \equiv \log \frac{\rho_h}{\rho_p}$$

"pseudo-energy"

$$w \equiv \sum_{n=0}^{\infty} \beta_n h_n$$

generalised Gibbs weight

$$F \equiv \log\left(1 + e^{-\epsilon}\right)$$

free energy density

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$$\Rightarrow \quad n(k) \equiv \left.\frac{\mathrm{d}F}{\mathrm{d}\epsilon}\right|_{\epsilon=\epsilon(k)} = \frac{e^{-\epsilon(k)}}{1 - e^{-\epsilon(k)}} = \frac{\rho_p(k)}{\rho_s(k)}$$

occupation function

Some more (maybe) familiar relations

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- Integrability: infinite number of conservation laws

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n diagonalises the dynamics:
Riemann invariant

Beyond Euler scale?

- Diffusive corrections

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- Extension to 2D problems?

GHD in a nutshell

	Homogeneous - Stationary		Inhomogeneous - Dynamical
Finite number of conserved quantities	Statistical Thermodynamics	⇒	Hydrodynamics
	⇓		⇓
Infinite number of conserved quantities	Generalized Gibbs Ensembles	⇒	Generalized Hydrodynamics