A Generalised Hydrodynamics approach to the Boussinesq equation: a prototypical example of 2D stationary soliton gas.

Integrable Systems seminar
University of Leeds

Thibault Bonnemain, 10th November 2023

Hydrodynamics in general

- Hydrodynamics is everywhere in physics:
  - Fluid dynamics (simple fluids Euler 1757)

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- Derived from an underlying microscopic model:

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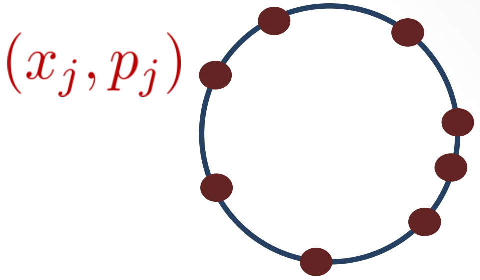
- Main ingredients:

$\Rightarrow$ local conservation laws $+$ propagation of local “equilibrium”

Conventional hydrodynamics: 1D fluid

- N particles on a circle of perimeter L



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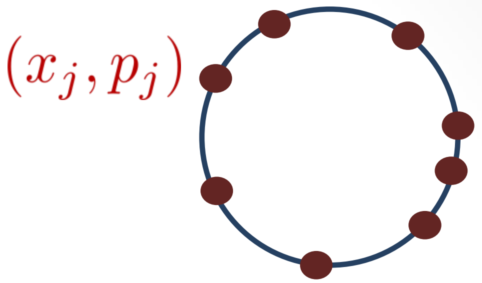


- N particles on a circle of perimeter L
- Conservation laws

$$N$$

Number of particle

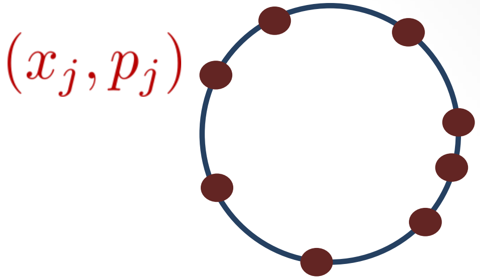
$$P = \sum_{j=1}^{N} p_j$$

Total momentum

$$E = \sum_{j=1}^{N} \frac{p_j^2}{2} + \sum_{i \neq j} \underbrace{V(x_i - x_j)}_{\text{Short range}}$$

Total energy

Conventional hydrodynamics: 1D fluid



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- Local densities

$$q_0(x) = \sum_{j=1}^{N} \delta(x - x_j)$$

$$q_1(x) = \sum_{j=1}^{N} \delta(x - x_j) p_j$$

$$q_2(x) = \sum_{j=1}^{N} \delta(x - x_j) \left[\frac{p_j^2}{2} + \sum_{i \neq j} V(x_i - x_j) \right]$$

so that

$$N = \int_0^L \mathrm{d}x \; q_0(x)$$

$$P = \int_0^L \mathrm{d}x \; q_1(x)$$

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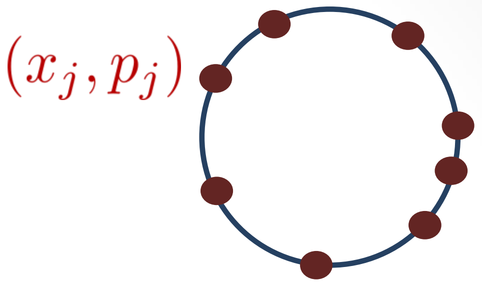

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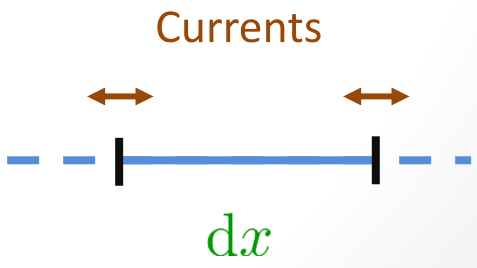

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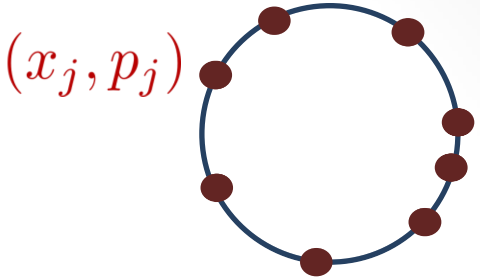

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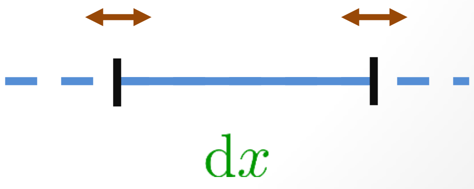

Functions on phase space or field operators

Local equilibrium

- Boltzmann 1868: micro-canonical ensemble in long time limit

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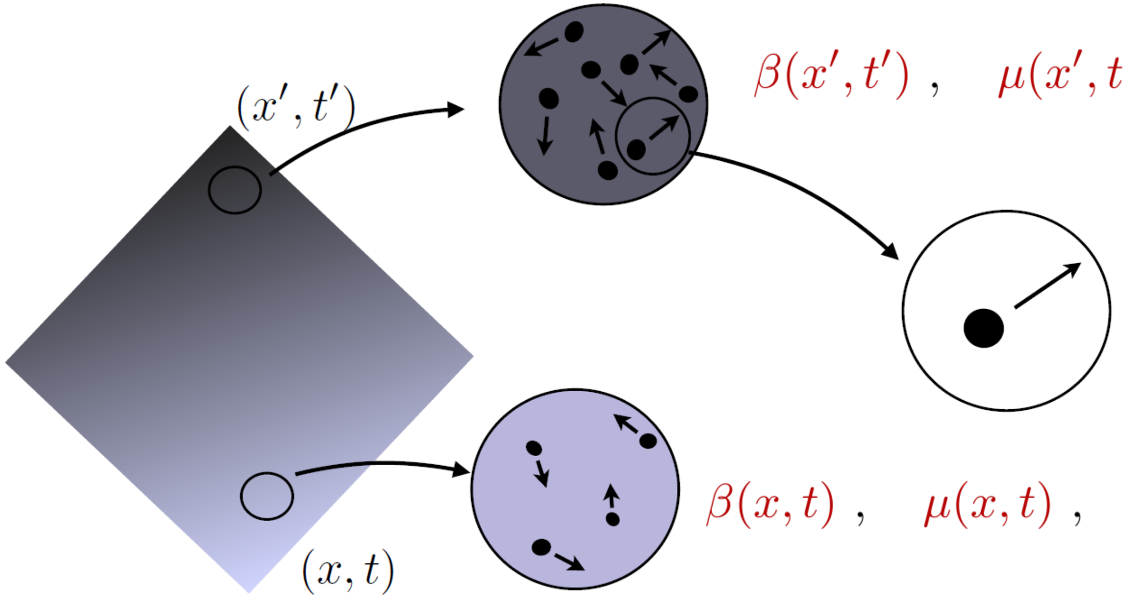
- Hydrodynamic principle: separation of scales and propagation of local GE

Macroscopic Mesoscopic (fluid cells) Microscopic



$\beta(x', t')$, $\mu(x', t')$, $\nu(x', t')$

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[Doyon: Lecture Notes (2020)]

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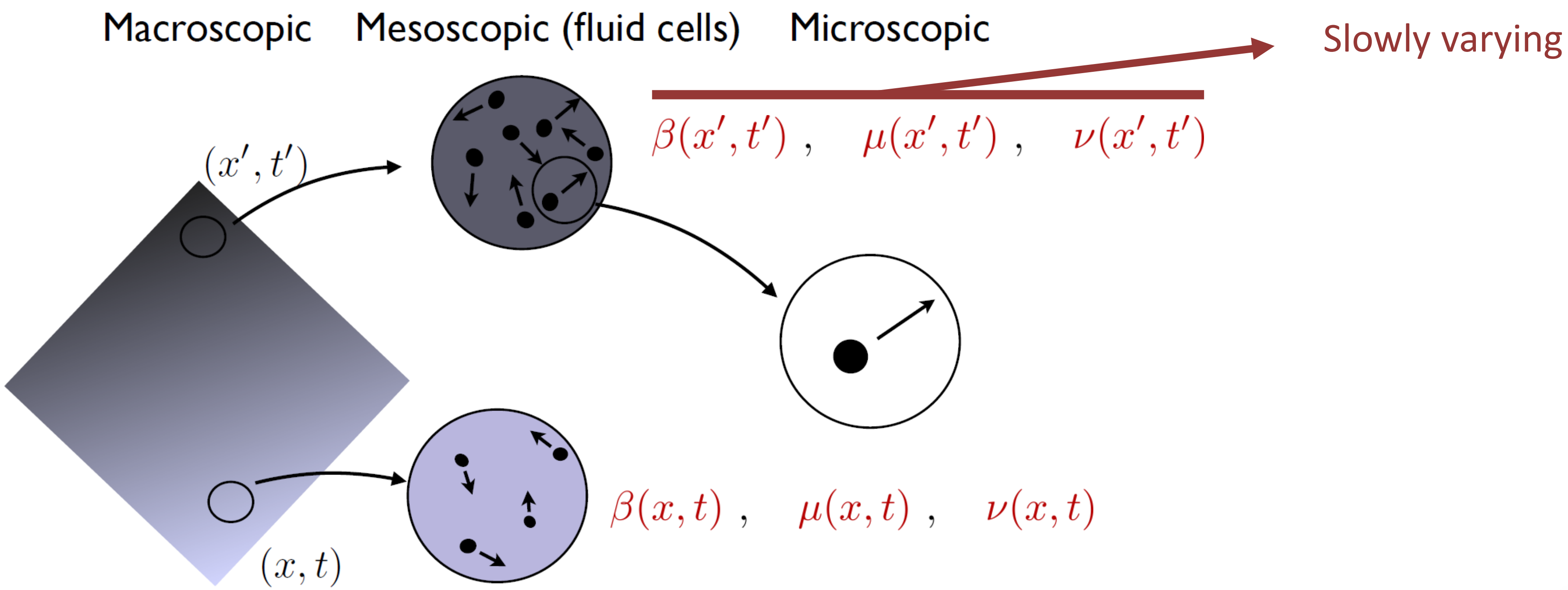

Macroscopic Mesoscopic (fluid cells) Microscopic

Slowly varying

$\beta(x',t')\ ,\quad \mu(x',t')\ ,\quad \nu(x',t')$

(x',t')

$\beta(x,t)\ ,\quad \mu(x,t)\ ,\quad \nu(x,t)$

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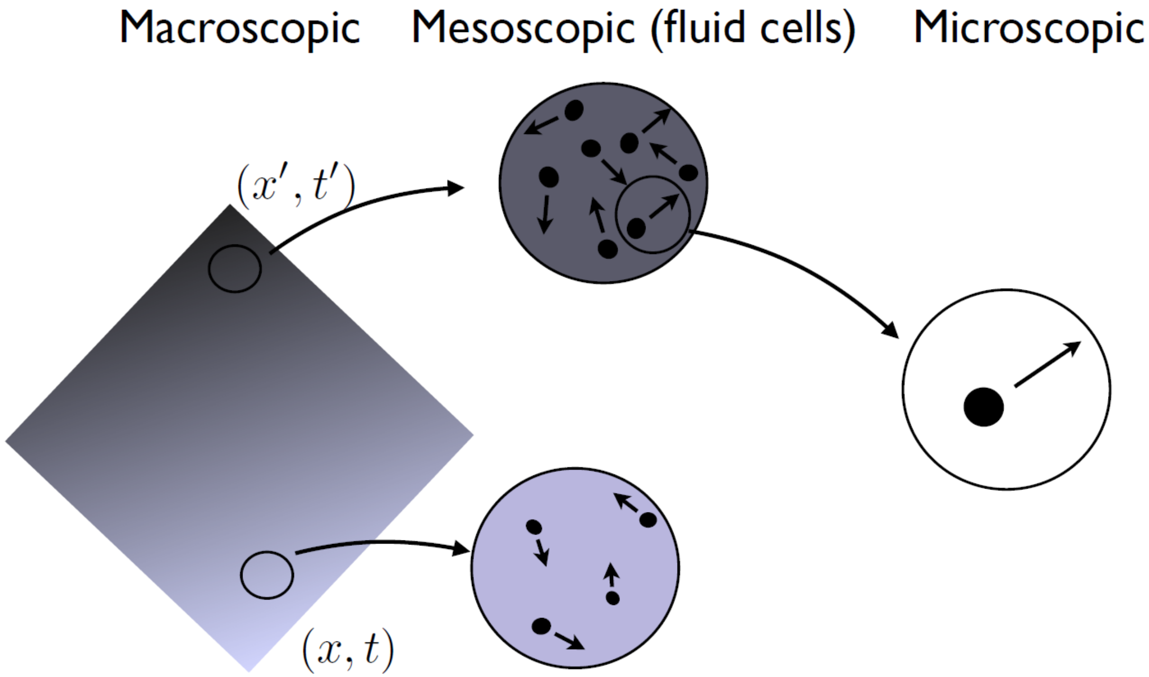

$\Rightarrow$ Local GE averages

$$\bar{q}_m(x,t) \quad \text{and} \quad \bar{j}_m(x,t)$$

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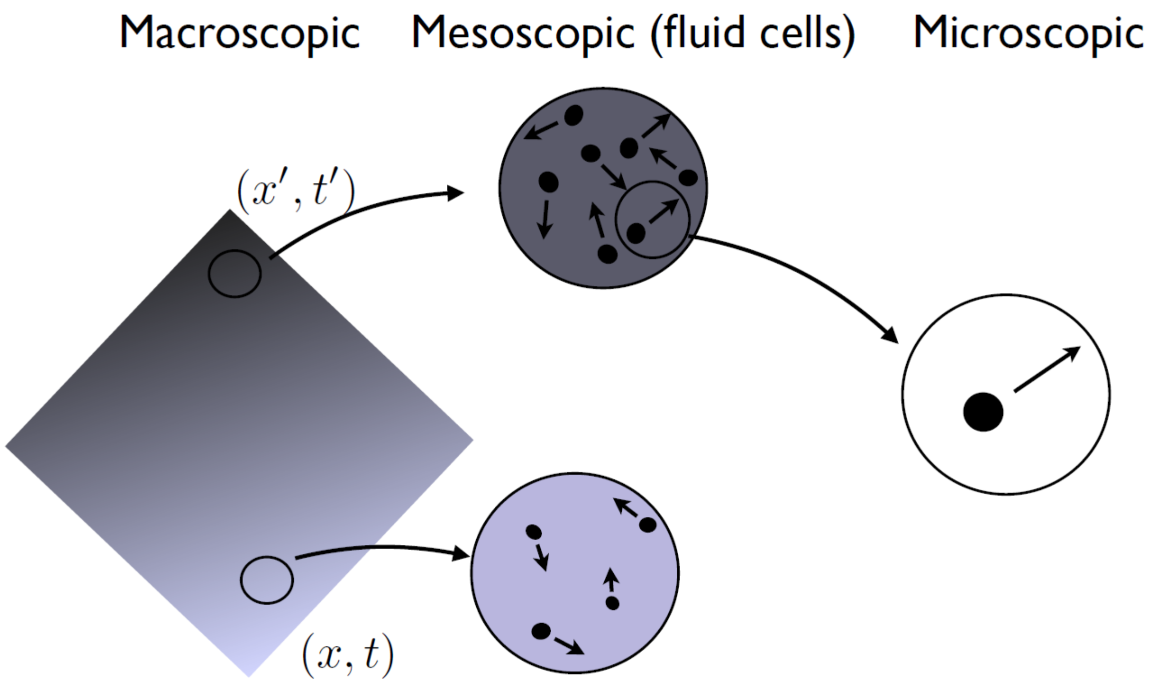

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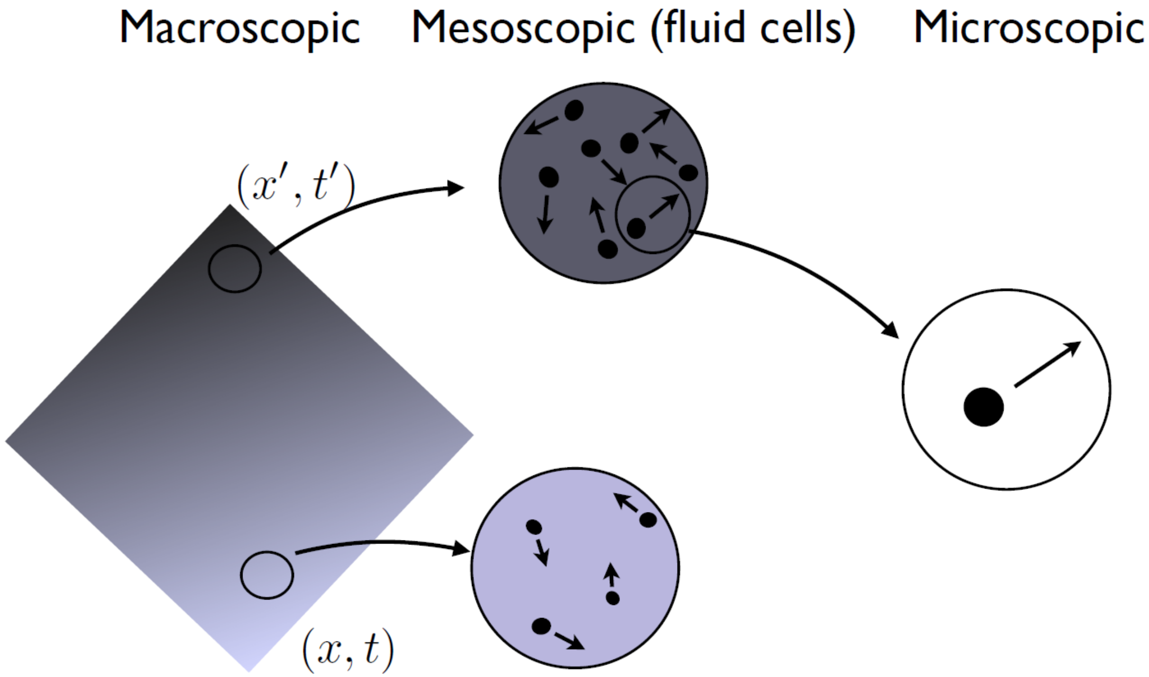

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Functions of $\{\bar{q}_n\}$'s !

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$$\triangleq \quad \text{Generalised Gibbs ensembles (GE):} \quad \rho \propto e^{-\sum_{n=0}^{\infty} \beta_n Q_n}$$

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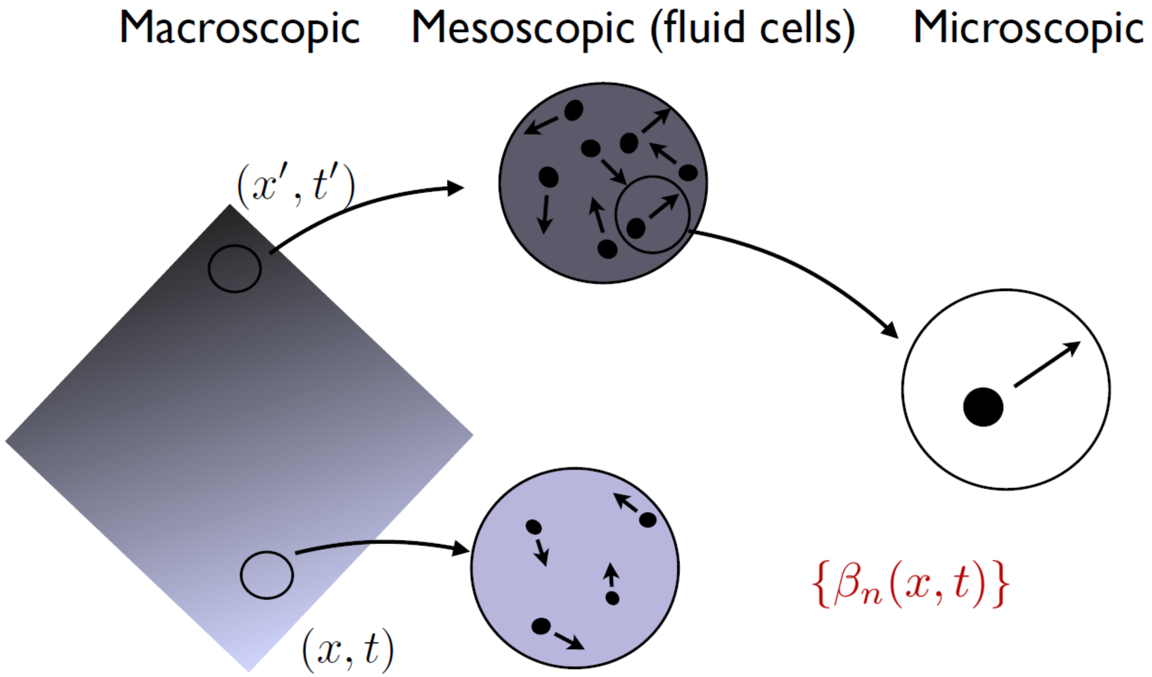

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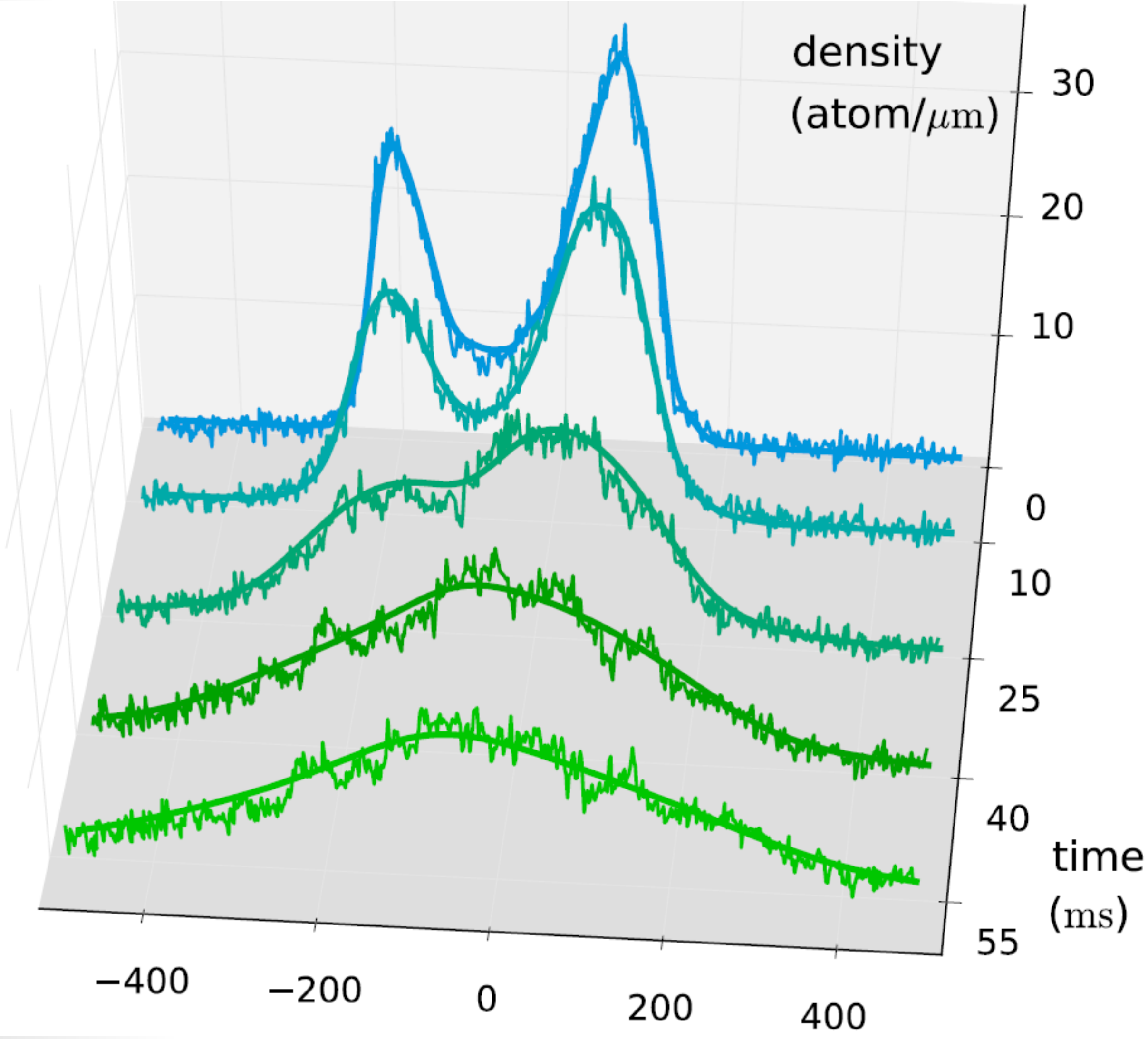
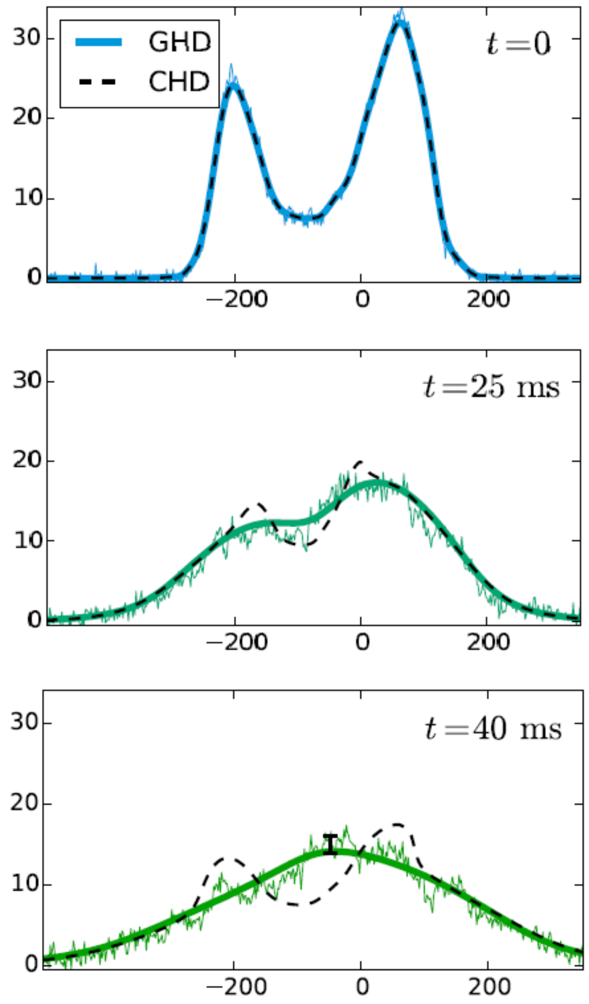
Example of GHD



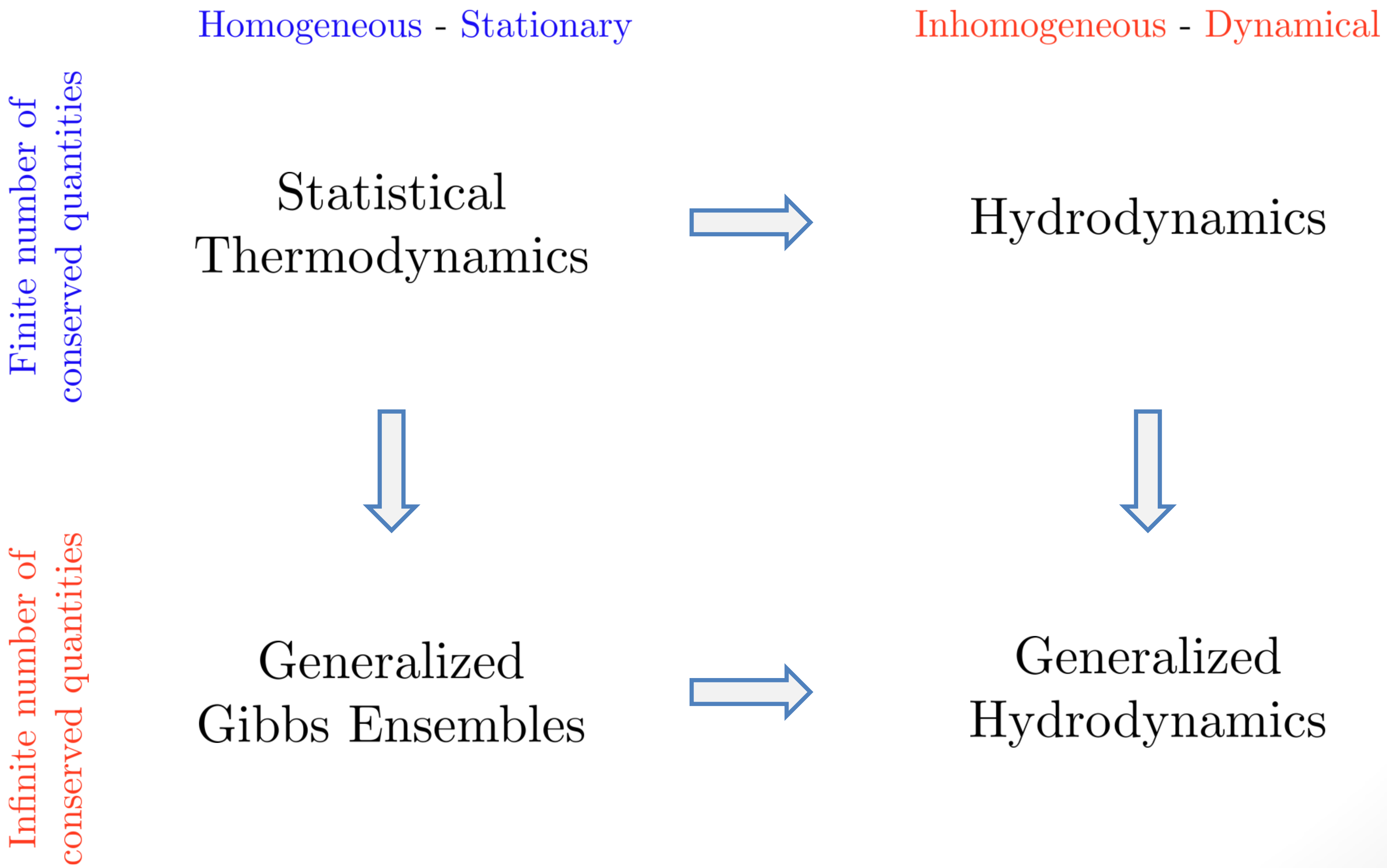


1D Bose gas is described by the Lieb-Liniger model

GHD experimentally observed on an atom chip

[Schemmer, Bouchoule, Doyon, Dubail (2019)]

GHD in a nutshell



Homogeneous - Stationary

Inhomogeneous - Dynamical

Finite number of conserved quantities

Statistical Thermodynamics

Hydrodynamics

Infinite number of conserved quantities

Generalized Gibbs Ensembles

Generalized Hydrodynamics

The Boussinesq equation as a stationary reduction of KP

- KP equation: integrable nonlinear dispersive PDE in (2+1)D

$$\left[u_t + 6\left(u^2\right)_x + u_{xxx}\right]_x + \sigma u_{yy} = 0 \, ,$$

where case $\sigma = -1$ refered to as KPI and $\sigma = 1$ as KPII.

The Boussinesq equation as a stationary reduction of KP

- (boosted) KP equation: integrable nonlinear dispersive PDE in(2+1)D

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- Stationary solutions solve

$$\sigma\left(u_{yy}-u_{xx}\right)+6\left(u^2\right)_x+u_{xxxx}=0\ ,$$

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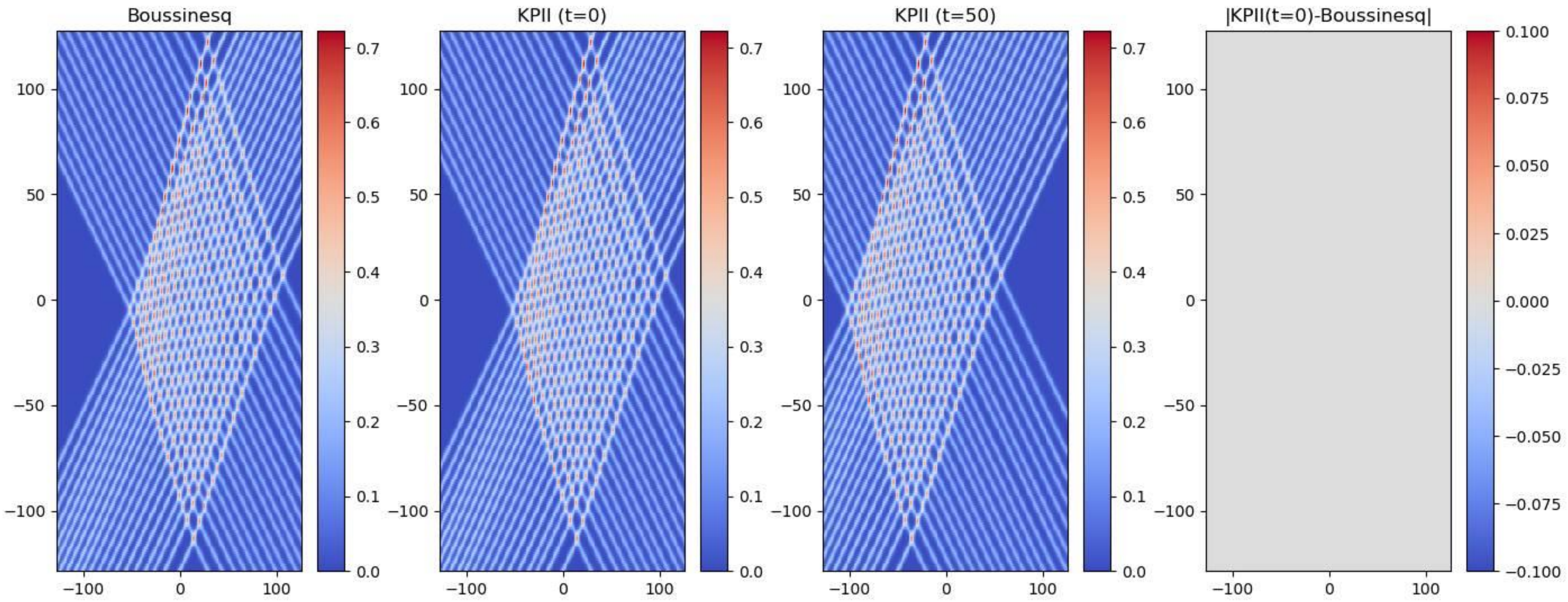
- Infinite set of conservation laws

$$\boxed{Q_n = \int \mathrm{d}x \ q_n(x,t) \ , \quad J_n = \int \mathrm{d}t \ j_n(x,t) \ , \quad \partial_t q_n + \partial_x j_n = 0 \ .}$$

Boussinesq vs (boosted) KP



N soliton solutions for the « good » Boussinesq equation

- $N-$soliton solution in terms of the $\tau-$function: $u_N(x,t) = [\log \tau(x,t)]_{xx}$

$$\tau(x,t) = 1 + \sum_{n=1}^{N} \sum_{{}_N C_n} a(i_1, i_2, \cdots, i_n) \exp\left[\theta_{i_1}(x,t) + \theta_{i_2}(x,t) + \cdots + \theta_{i_n}(x,t)\right] \;,$$

with

$$\theta_j(x,t) = \eta_i \left(x - \epsilon_i t \sqrt{1-\eta_i^2} - x_i^0 \right) \;,$$

$$a(i_1, i_2, \cdots, i_n) = \prod_{k<l}^{n} \exp \varphi_{i_k i_l} \;,$$

$$\varphi_{ij} = \log \frac{\left(\epsilon_i \sqrt{1-\eta_i^2} - \epsilon_j \sqrt{1-\eta_j^2}\right)^2 - 3(\eta_i - \eta_j)^2}{\left(\epsilon_i \sqrt{1-\eta_i^2} - \epsilon_j \sqrt{1-\eta_j^2}\right)^2 - 3(\eta_i + \eta_j)^2} \;.$$

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$N-$soliton solution completely specified by the triplets $\{\eta_i, x_i^0, \epsilon_i\}_{i=1}^N$

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φ_{ij}^{+} if $\epsilon_i \epsilon_j = 1$

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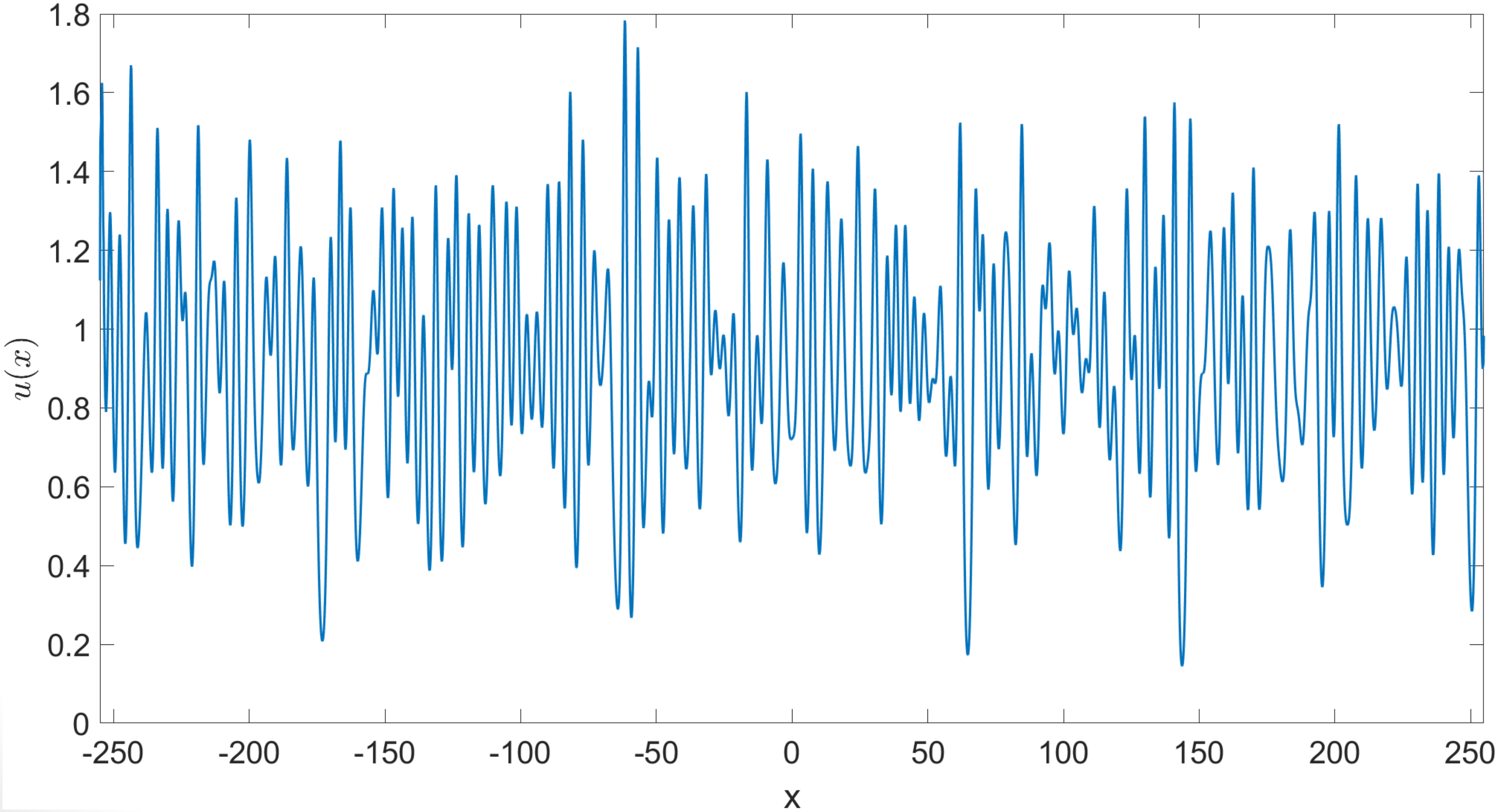
φ_{ij}^{-} if $\epsilon_i \epsilon_j = -1$

GHD from scattering theory

- Soliton gas: random solution to Boussinesq that can be well-described on a large interval by some $N-$soliton ensemble.



Single realisation of a soliton gas

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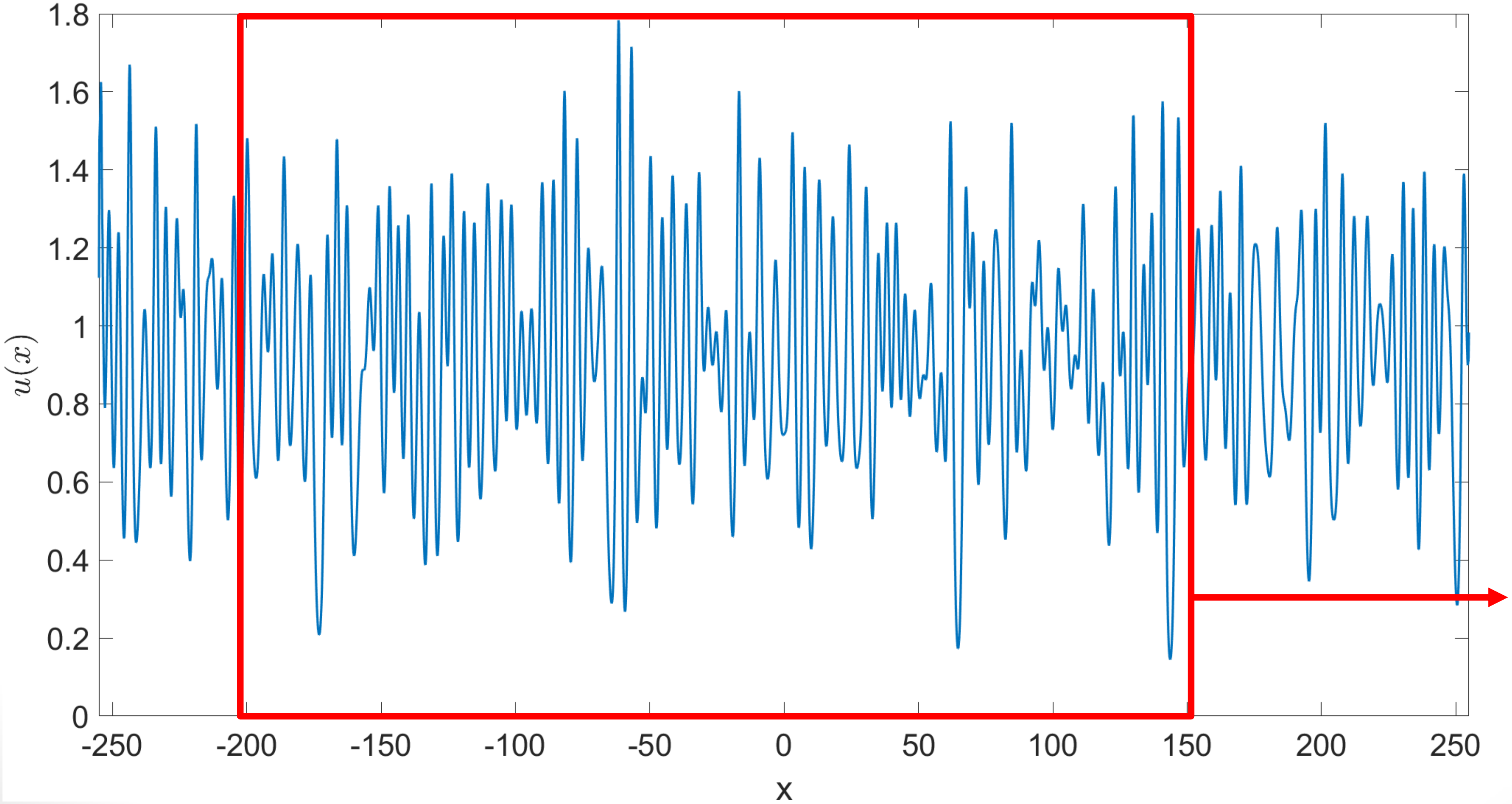

Single realisation of a soliton gas

Fluid cell of size L characterised by local GGE

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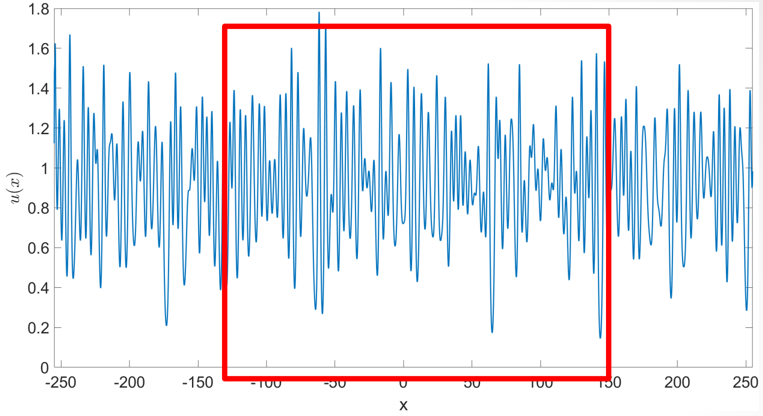
- Multi-soliton solution


N solitons in L

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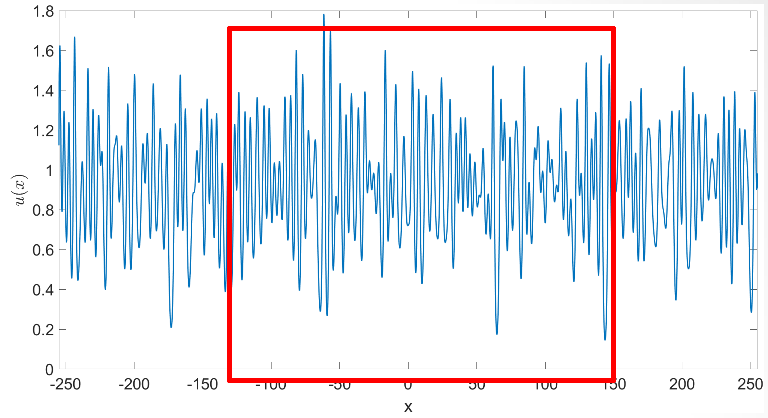

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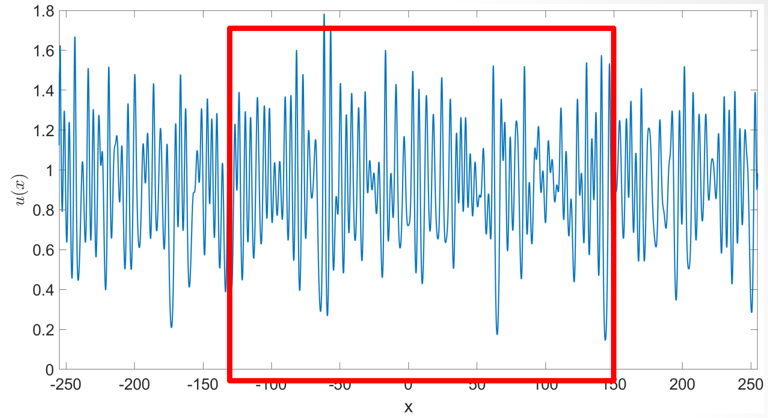

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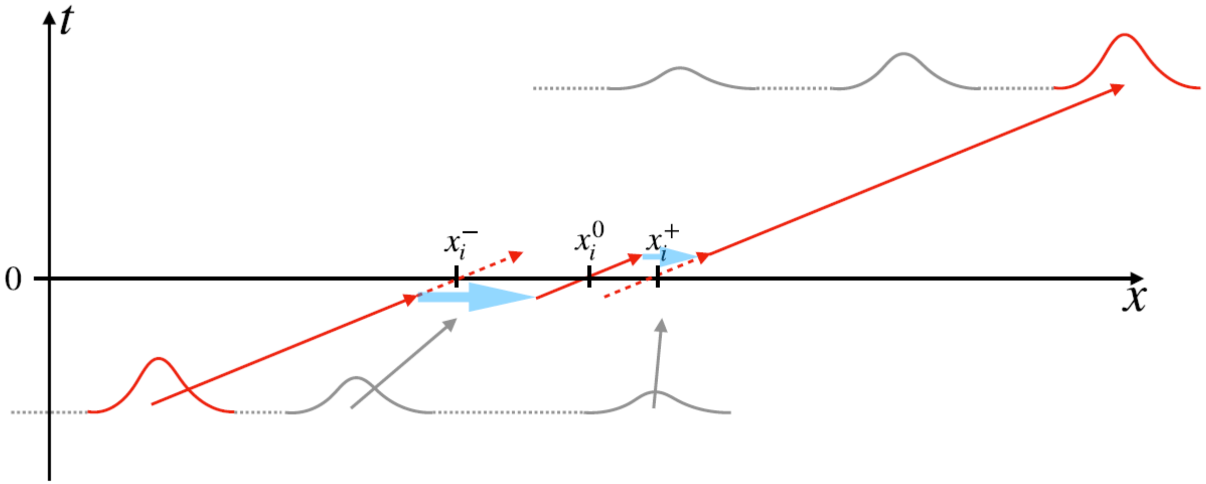
Action coordinate

Angle coordinate



Scattering is elastic and
2-body factorisable

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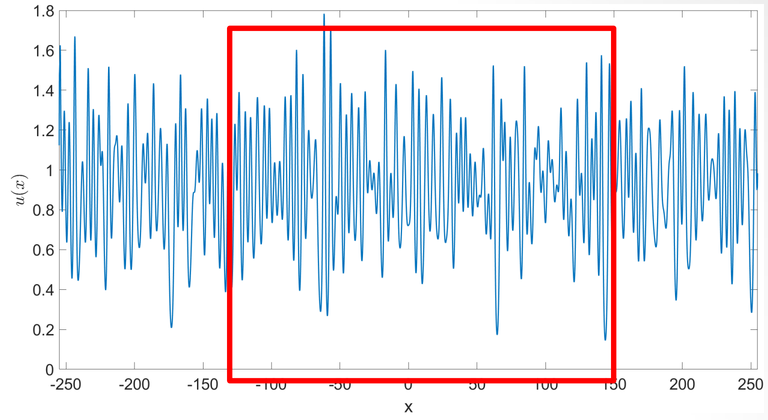
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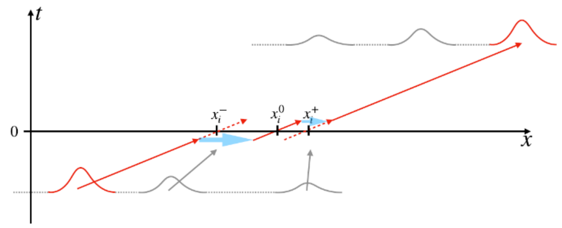
- Relation between asymptotic states given by scattering shift

$$x_i^+ - x_i^- = \sum_{j\neq i} \Delta_{ij} , \quad \Delta_{ij} = \begin{cases} \operatorname{sgn}(\eta_i - \eta_j)\dfrac{\varphi_{ij}^+}{\eta_i} & \text{if } \epsilon_i\epsilon_j = 1 \\ -\dfrac{\varphi_{ij}^-}{\epsilon_i\eta_i} & \text{if } \epsilon_i\epsilon_h = -1 \end{cases}$$



Thermodynamics

- $N-$soliton partition function can be formally written as

$$\mathcal{Z}_N = \int \mathcal{D}[u_N] \exp\left(\underline{S[u_N]} - \underline{W[u_N]}\right) .$$

Entropy

Generalised Gibbs weight $W = \sum_k \beta_k Q_k$

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Constraint / Entropy

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- Assume solitons are point particles of velocity v_i and position $X_i^t = x_i^t + v_i t$.

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- Let i be the leftmost soliton $(x_i^0 = 0)$

$$0 \quad = \quad x_i^{\text{left}} + \frac{1}{\eta_i} \sum_{j=i+1}^{N} \varphi_{ij}^{+} \; .$$

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$$\underbrace{0}_{\text{Position at t=0}} = \overbrace{x_i^{\text{left}}}^{\text{Asymptotic position } x_i^-} + \underbrace{\frac{1}{\eta_i} \sum_{j=i+1}^{N} \varphi_{ij}^{+}}_{\text{Shifts from faster solitons}} .$$

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- Let i be the rightmost soliton $(x_i^0 = L)$

$$L = x_i^{\text{right}} - \frac{1}{\eta_i} \left[\sum_{j=M+1}^{i-1} \varphi_{ij}^{+} + \sum_{j=1}^{M} \varphi_{ij}^{-} \right] \ .$$

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⇒

Asymptotic space in terms of real space

$$\begin{aligned} L_i^{\text{r}} &\equiv x_i^{\text{right}} - x_i^{\text{left}} \\ &= L + \frac{1}{\eta_i} \left[\sum_{j=1}^{M} \varphi_{ij}^{-} + \sum_{j=M+1,\ j \neq i}^{N} \varphi_{ij}^{+} \right] \end{aligned}$$

Asymptotic space density

- Let $L^{\mathrm{r}}_N(\eta)$ interpolate L^{r}_i

$$\mathcal{K}^{\mathrm{r}}_N(\eta) \equiv \frac{L^{\mathrm{r}}_N(\eta)}{L} = 1 + \frac{1}{L\eta}\left[\sum_{j=1}^{M} \varphi^{-}(\eta,\eta_j) + \sum_{j=M+1,\ j\neq i}^{N} \varphi^{+}(\eta,\eta_j)\right] .$$

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$$\underline{\mathcal{K}^{\mathrm{r}}(\eta)} = 1 + \frac{1}{\eta}\left[\int_{\Gamma_{\mathrm{l}}} \mathrm{d}\mu \ \underline{\rho^{\mathrm{l}}(\mu)}\varphi^-(\eta,\mu) + \int_{\Gamma_{\mathrm{r}}} \mathrm{d}\mu \ \underline{\rho^{\mathrm{r}}(\mu)}\varphi^+(\eta,\mu)\right] .$$

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$$\mathrm{d}x_{\mathrm{r}}^{-}(\eta) = \mathcal{K}^{\mathrm{r}}(\eta)\mathrm{d}x$$

change of metric due to interactions

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$$\int_{\mathbb{R}^N} \prod_{i=1}^{N} \mathrm{d}x_i^- \chi\left(u_N(x, t=0) < \varepsilon_x,\ x \notin [0, L]\right) \approx \prod_{i=1}^{N} \left(\int_{x_i^{\mathrm{left}}}^{x_i^{\mathrm{right}}} \mathrm{d}x^- \right)$$

$$= L^N \prod_{i=1}^{M} \mathcal{K}^{\mathrm{l}}(\eta_i) \prod_{i=M+1}^{N} \mathcal{K}^{\mathrm{r}}(\eta_i)$$

Thermodynamic equilibrium

- Large deviations theory

[Varadhan (1966), Touchette (2009)]

$$\mathcal{Z}_N \asymp \exp\left(-L\mathcal{F}^{\mathrm{MF}}[\bar{\rho}^{\mathrm{l}}(\eta), \bar{\rho}^{\mathrm{r}}(\eta)]\right)$$

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$$\begin{aligned}\mathcal{F}^{\mathrm{MF}}[\rho_{\mathrm{l}}(\eta), \rho_{\mathrm{r}}(\eta)] = &\int_{\Gamma_{\mathrm{l}}} \mathrm{d}\eta \rho_{\mathrm{l}}(\eta) \left[w_{\mathrm{l}}(\eta) - 1 + \nu - \log \frac{\eta}{2\pi\sqrt{1-\eta^2}} - \log \mathcal{K}_{\mathrm{l}}(\eta) + \log \rho_{\mathrm{l}}(\eta)\right] \\ &+ \int_{\Gamma_{\mathrm{r}}} \mathrm{d}\eta \rho_{\mathrm{r}}(\eta) \left[w_{\mathrm{r}}(\eta) - 1 + \nu - \log \frac{\eta}{2\pi\sqrt{1-\eta^2}} - \log \mathcal{K}_{\mathrm{r}}(\eta) + \log \rho_{\mathrm{r}}(\eta)\right]\end{aligned}$$

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[Sanov (1961)]

$$\mathcal{F}^{\mathrm{MF}}[\rho_{\mathrm{l}}(\eta), \rho_{\mathrm{r}}(\eta)] = \int_{\Gamma_{\mathrm{l}}} \mathrm{d}\eta\, \rho_{\mathrm{l}}(\eta) \left[\overbrace{w_{\mathrm{l}}(\eta)}^{\text{Gibbs weights}} - 1 + \nu - \overbrace{\log \frac{\eta}{2\pi\sqrt{1-\eta^2}}}^{\text{Jacobian}} - \log \mathcal{K}_{\mathrm{l}}(\eta) + \overbrace{\log \rho_{\mathrm{l}}(\eta)}^{\text{Configuration entropy}}\right]$$
$$+ \int_{\Gamma_{\mathrm{r}}} \mathrm{d}\eta\, \rho_{\mathrm{r}}(\eta) \left[w_{\mathrm{r}}(\eta) \underbrace{- 1 + \nu -}_{\text{Prefactor}} \log \frac{\eta}{2\pi\sqrt{1-\eta^2}} - \underbrace{\log \mathcal{K}_{\mathrm{r}}(\eta)}_{\text{Constraint}} + \log \rho_{\mathrm{r}}(\eta)\right]$$

$$\nu = \log\left[\gamma^{\gamma}(1-\gamma)^{1-\gamma}\right]$$

Yang-Yang type system

- Minimisation condition for the free energy functional

$$\begin{cases} \varepsilon_{\mathrm{l}}(\eta) = w_{\mathrm{l}}(\eta) + \nu + \log|v(\eta)| - \displaystyle\int_{\Gamma_{\mathrm{l}}} \frac{\mathrm{d}\mu}{2\pi}\, \varphi^{+}(\eta,\mu) e^{-\varepsilon_{\mathrm{l}}(\mu)} - \displaystyle\int_{\Gamma_{\mathrm{r}}} \frac{\mathrm{d}\mu}{2\pi} \varphi^{-}(\eta,\mu) e^{-\varepsilon_{\mathrm{r}}(\mu)} \\ \varepsilon_{\mathrm{r}}(\eta) = w_{\mathrm{r}}(\eta) + \nu + \log|v(\eta)| - \displaystyle\int_{\Gamma_{\mathrm{r}}} \frac{\mathrm{d}\mu}{2\pi}\, \varphi^{+}(\eta,\mu) e^{-\varepsilon_{\mathrm{r}}(\mu)} - \displaystyle\int_{\Gamma_{\mathrm{l}}} \frac{\mathrm{d}\mu}{2\pi} \varphi^{-}(\eta,\mu) e^{-\varepsilon_{\mathrm{l}}(\mu)} \end{cases} .$$

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- Entropy of the soliton gas $\mathcal{S} = \mathcal{W} - \mathcal{F}$

$$\mathcal{S} = \int_{\Gamma_{\mathrm{l}}} \mathrm{d}\eta\ \rho_{\mathrm{l}}(\eta)\left[1 - \log n_{\mathrm{l}}(\eta) - \nu - \log|v(\eta)|\right] + \int_{\Gamma_{\mathrm{r}}} \mathrm{d}\eta\ \rho_{\mathrm{r}}(\eta)\left[1 - \log n_{\mathrm{r}}(\eta) - \nu - \log|v(\eta)|\right]$$

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- Integrability: infinite number of conservation laws

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$$\bar{j}_n(x,t) = \int_{\Gamma_\mathrm{l}} \mathrm{d}\eta\ \rho_\mathrm{l}(\eta;x,t) h_n^\mathrm{l}(\eta) v_\mathrm{l}^\mathrm{eff}(\eta;x,t) + \int_{\Gamma_\mathrm{r}} \mathrm{d}\eta\ \rho_\mathrm{r}(\eta;x,t) h_n^\mathrm{r}(\eta) v_\mathrm{r}^\mathrm{eff}(\eta;x,t)$$

From thermodynamics to hydrodynamics

[Based on: Doyon, Spohn, Yoshimura (2017)]

- Asymptotic dynamics

$$x_i^-(t) = x_i^-(0) + v(\eta_i)t$$

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$$\begin{cases} v_{\mathrm{l}}^{\mathrm{eff}}(\eta) = -v(\eta) - \dfrac{1}{\eta}\left[\displaystyle\int_{\Gamma_{\mathrm{l}}} \mathrm{d}\mu\ \varphi^+(\eta,\mu)\rho_{\mathrm{l}}(\mu)[v_{\mathrm{l}}^{\mathrm{eff}}(\eta) - v_{\mathrm{l}}^{\mathrm{eff}}(\mu)] + \int_{\Gamma_{\mathrm{r}}} \mathrm{d}\mu\ \varphi^-(\eta,\mu)\rho_{\mathrm{r}}(\mu)[v_{\mathrm{l}}^{\mathrm{eff}}(\eta) - v_{\mathrm{r}}^{\mathrm{eff}}(\mu)]\right] \\ v_{\mathrm{r}}^{\mathrm{eff}}(\eta) = v(\eta) - \dfrac{1}{\eta}\left[\displaystyle\int_{\Gamma_{\mathrm{r}}} \mathrm{d}\mu\ \varphi^+(\eta,\mu)\rho_{\mathrm{r}}(\mu)[v_{\mathrm{r}}^{\mathrm{eff}}(\eta) - v_{\mathrm{r}}^{\mathrm{eff}}(\mu)] + \int_{\Gamma_{\mathrm{l}}} \mathrm{d}\mu\ \varphi^-(\eta,\mu)\rho_{\mathrm{l}}(\mu)[v_{\mathrm{r}}^{\mathrm{eff}}(\eta) - v_{\mathrm{l}}^{\mathrm{eff}}(\mu)]\right] \end{cases}$$

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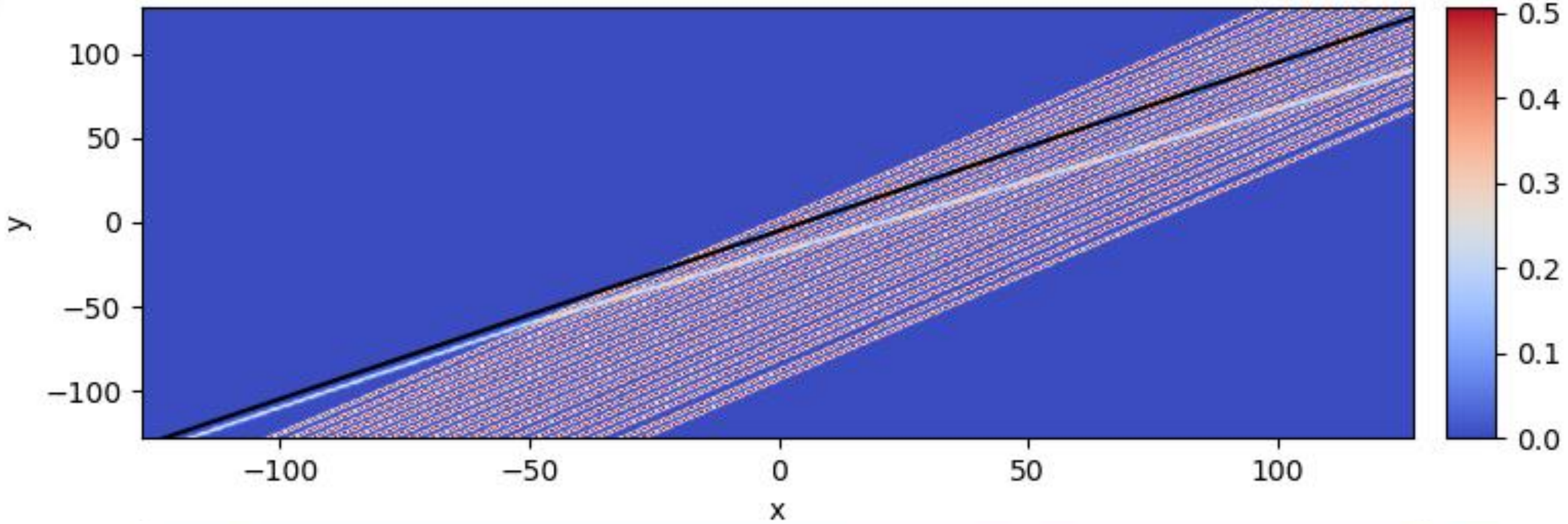
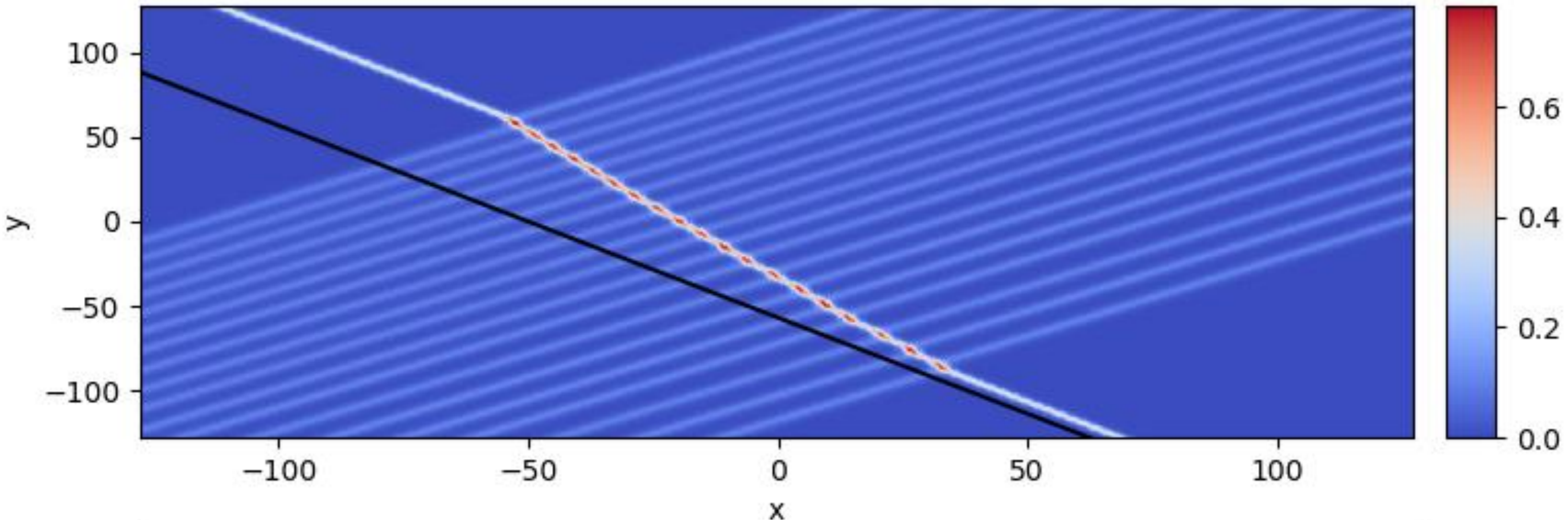
- Continuity equation for the DOS's

$$\boxed{\partial_t \rho_\cdot(\eta; x, t) + \partial_x\left[\rho_\cdot(\eta; x, t)v^{\mathrm{eff}}_\cdot(\eta; x, t)\right] = 0}$$

Simulations





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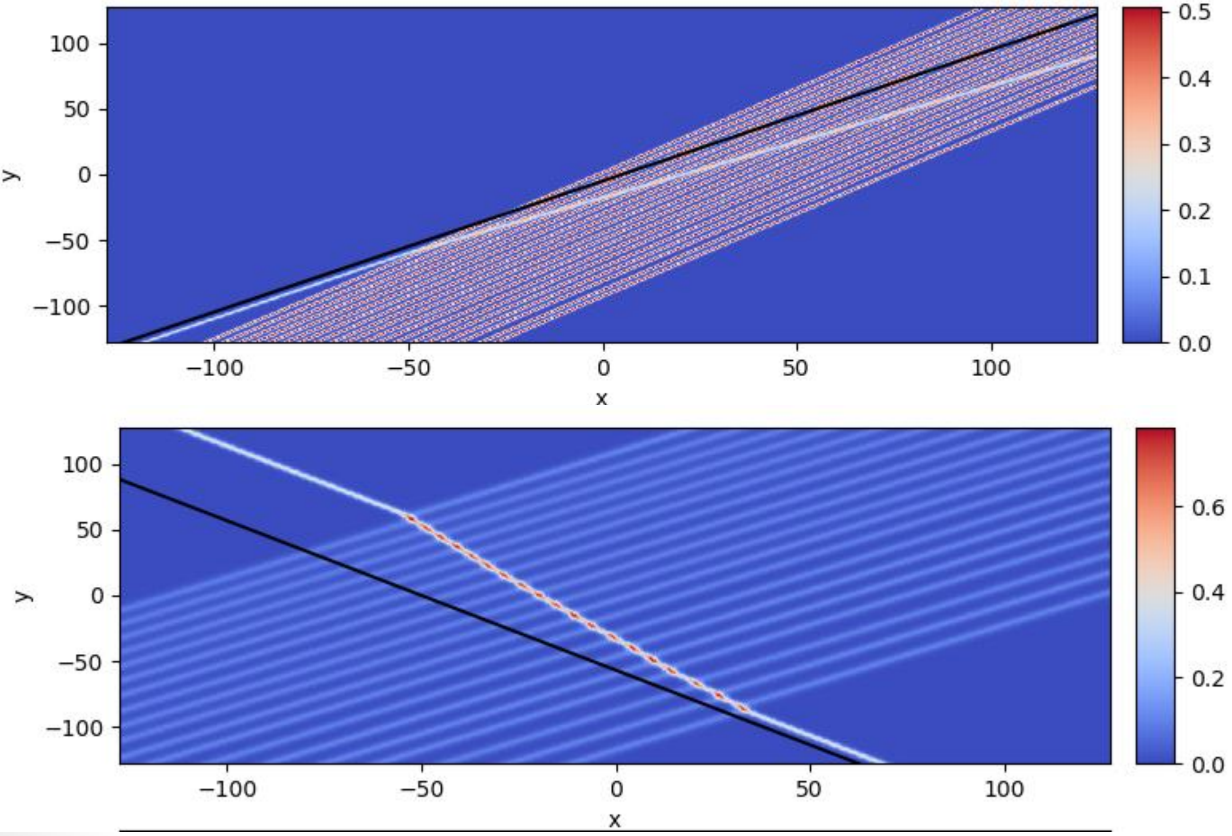

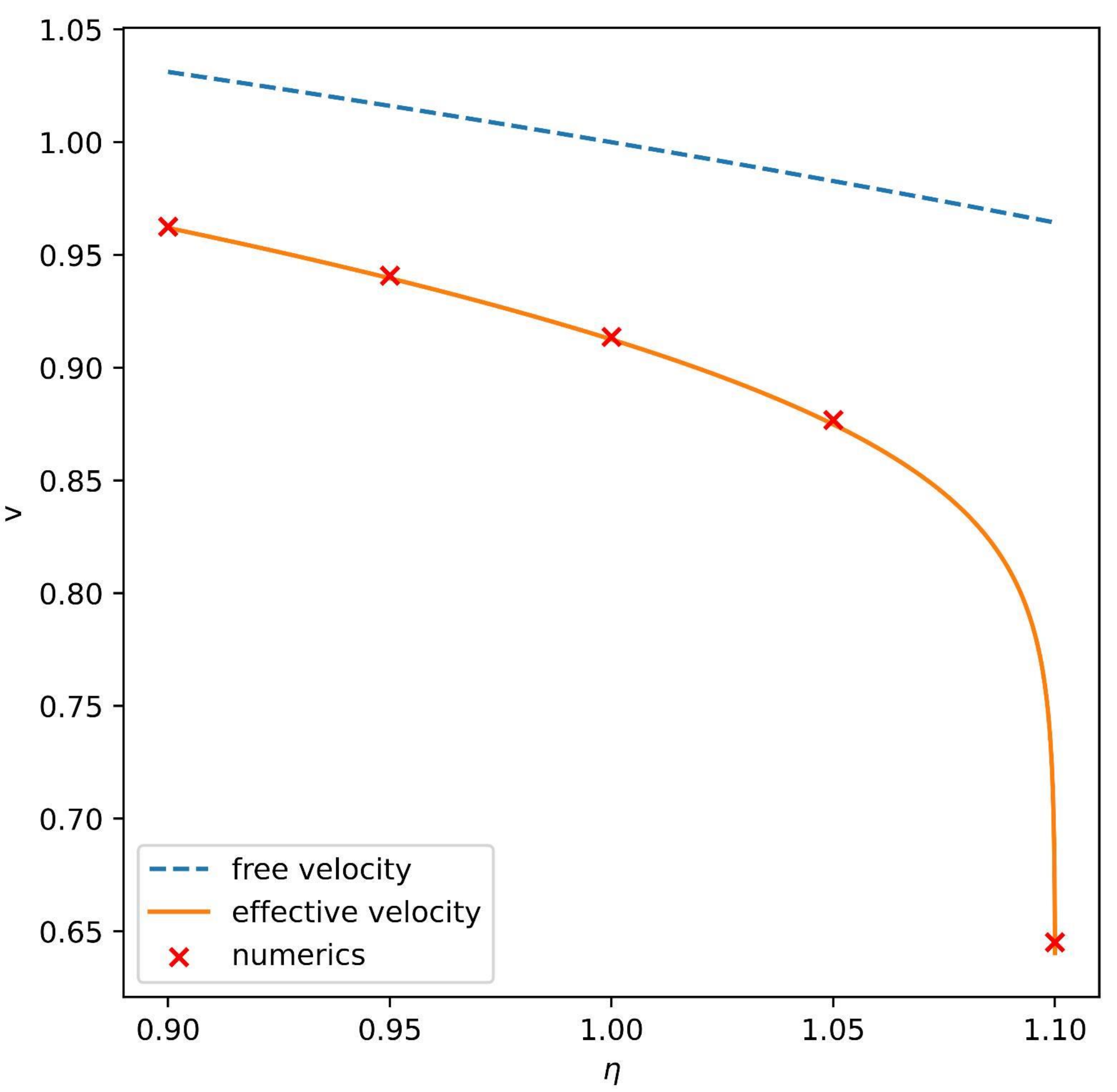

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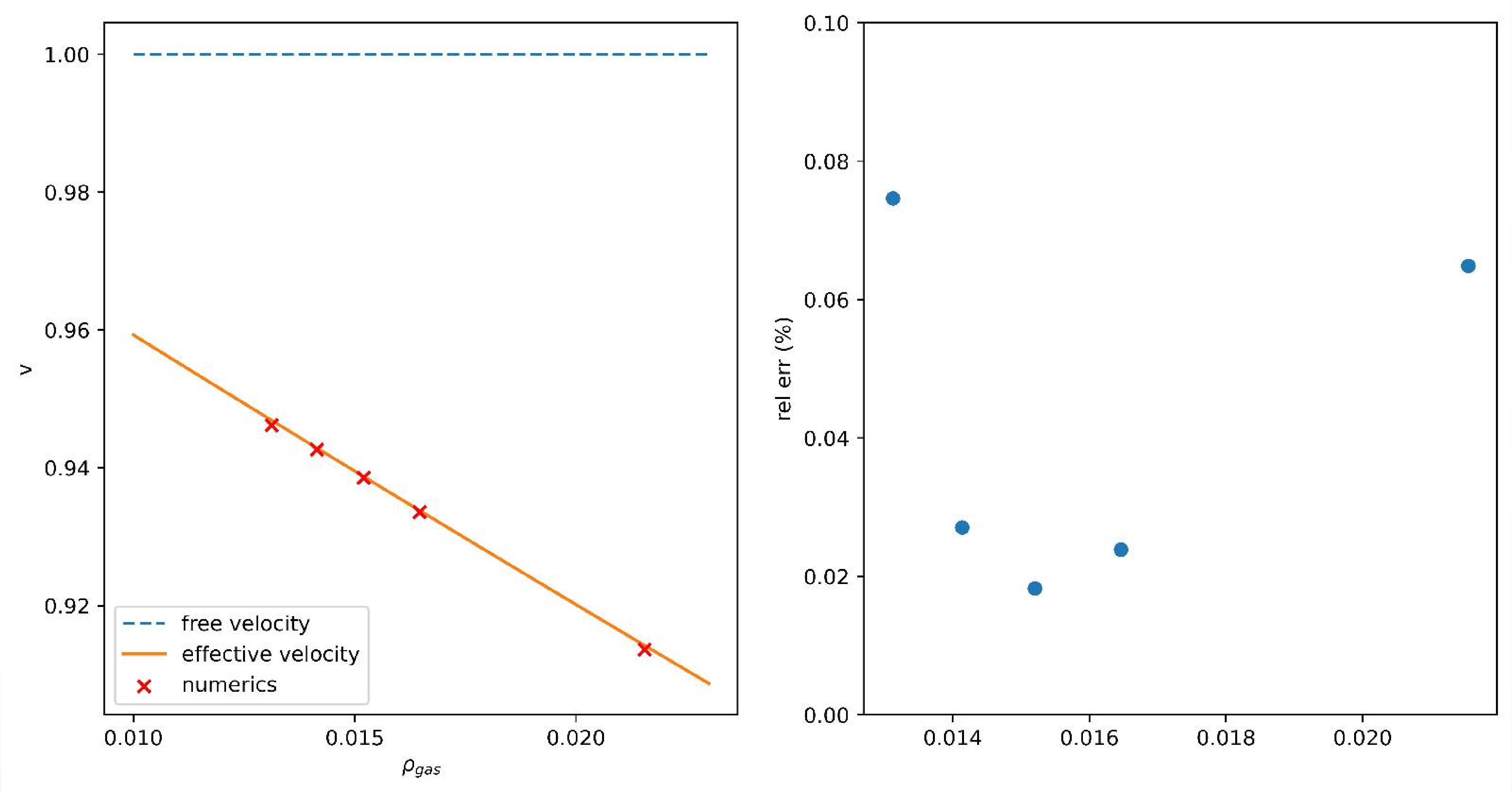

Concluding remarks and perspectives

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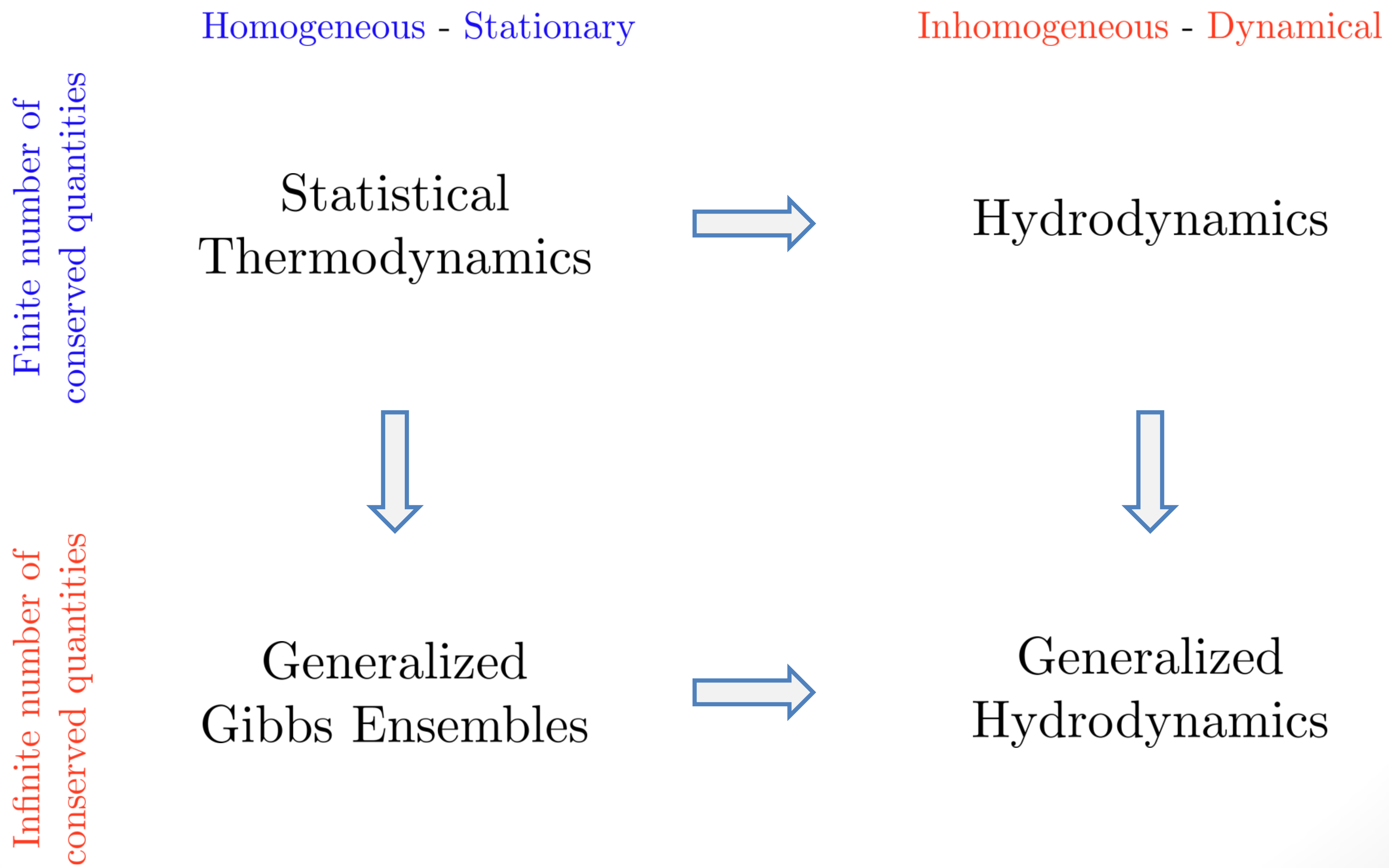
- Generic way to study integrable models in (d+1)D that involve "solitons" of co-dimension 1?

GHD in a nutshell



Homogeneous - Stationary

Inhomogeneous - Dynamical

Finite number of conserved quantities

Statistical Thermodynamics

Hydrodynamics

Infinite number of conserved quantities

Generalized Gibbs Ensembles

Generalized Hydrodynamics