LONDON MATHEMATICAL SOCIETY
EST. 1865

MCNP
Mathematics of Complex and Nonlinear Phenomena

KING'S College LONDON

Towards a (2+1)D Generalised Hydrodynamics

ISIN2023
Northumbria University

Thibault Bonnemain, 28th April 2023

[Based on preliminary work with Benjamin Doyon]

Generalised hydrodynamics and hydrodynamics in general

- Hydrodynamics is everywhere in physics:
    - Fluid dynamics (simple fluids Euler 1757)

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- Derived from an underlying microscopic model:

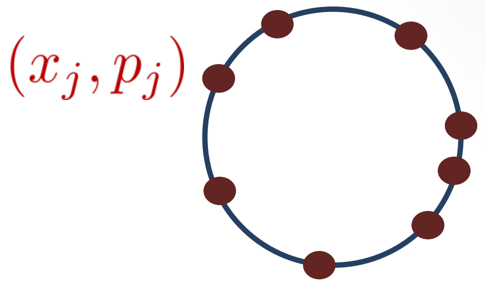
$\Rightarrow$ field theories or many-particle systems

- Main ingredients:

$\Rightarrow$ local conservation laws $+$ propagation of local "equilibrium"

Conventional hydrodynamics: 1D fluid

- N particles on a circle of perimeter L



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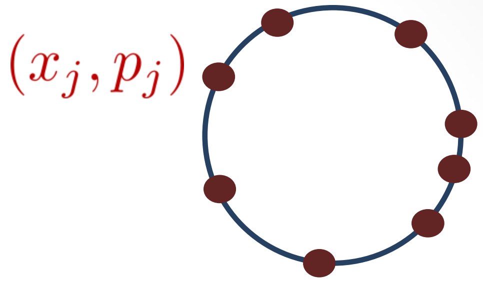


- N particles on a circle of perimeter L
- Conservation laws

$$N$$

Number of particle

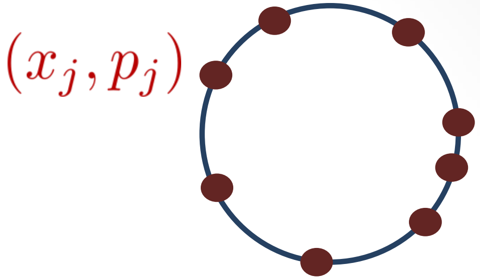
$$P = \sum_{j=1}^{N} p_j$$

Total momentum

$$E = \sum_{j=1}^{N} \frac{p_j^2}{2} + \sum_{i \neq j} \underbrace{V(x_i - x_j)}_{\text{Short range}}$$

Total energy

Conventional hydrodynamics: 1D fluid



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- Local densities

$$q_0(x) = \sum_{j=1}^{N} \delta(x - x_j)$$

$$q_1(x) = \sum_{j=1}^{N} \delta(x - x_j) p_j$$

$$q_2(x) = \sum_{j=1}^{N} \delta(x - x_j) \left[\frac{p_j^2}{2} + \sum_{i \neq j} V(x_i - x_j) \right]$$

so that

$$N = \int_0^L \mathrm{d}x \; q_0(x)$$

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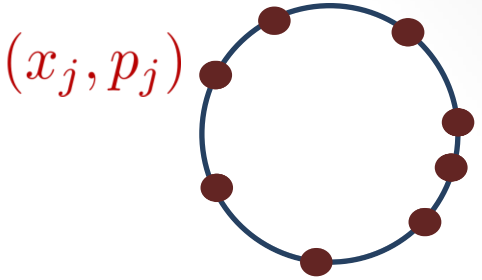

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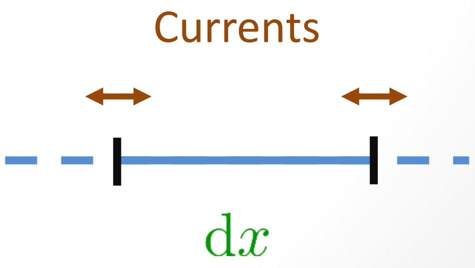
$$\partial_t q_m(x,t) + \partial_x j_m(x,t) = 0 \; , \quad m = 0, \; 1, \; 2.$$



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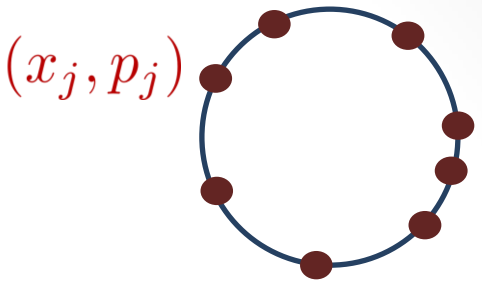

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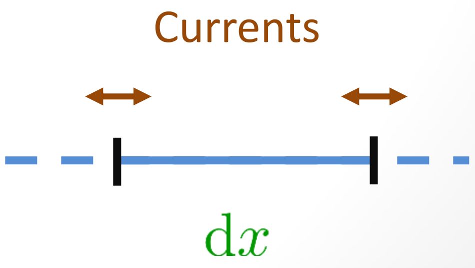

Functions on phase space or field operators

Local equilibrium

- Boltzmann 1868: micro-canonical ensemble in long time limit

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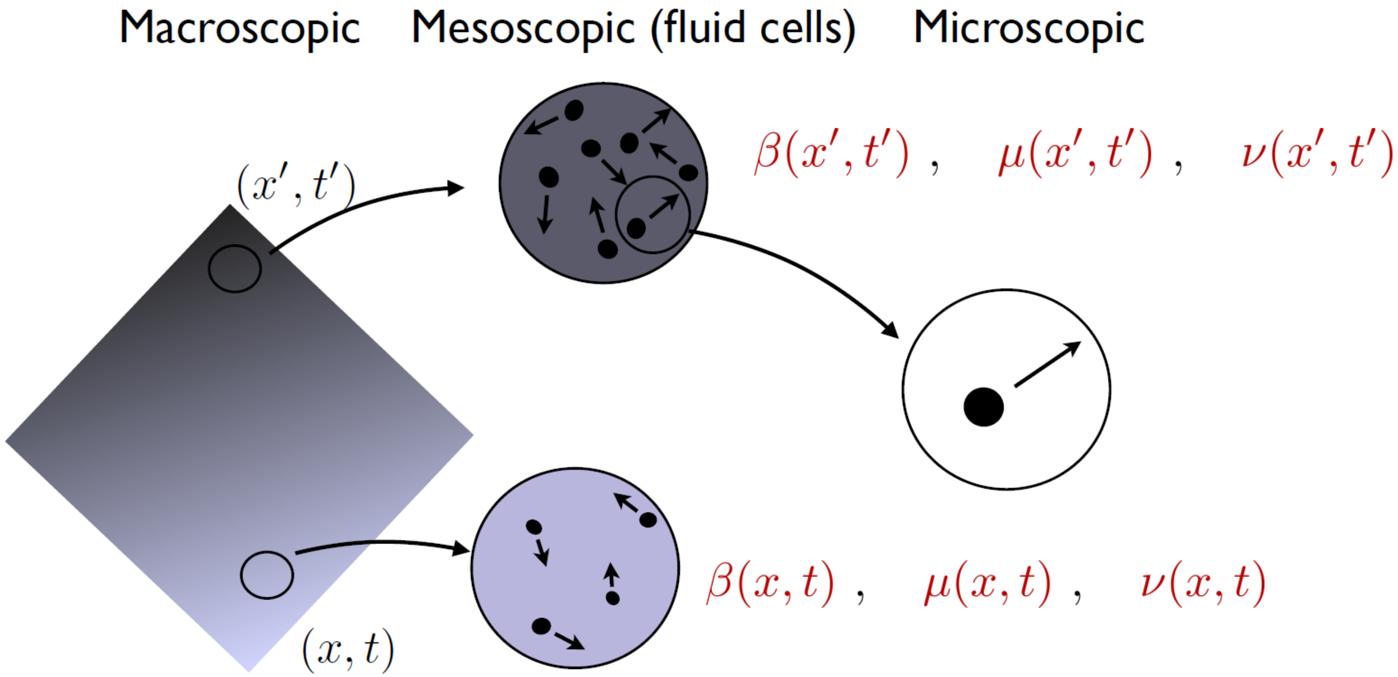

[Doyon: Lecture Notes (2020)]

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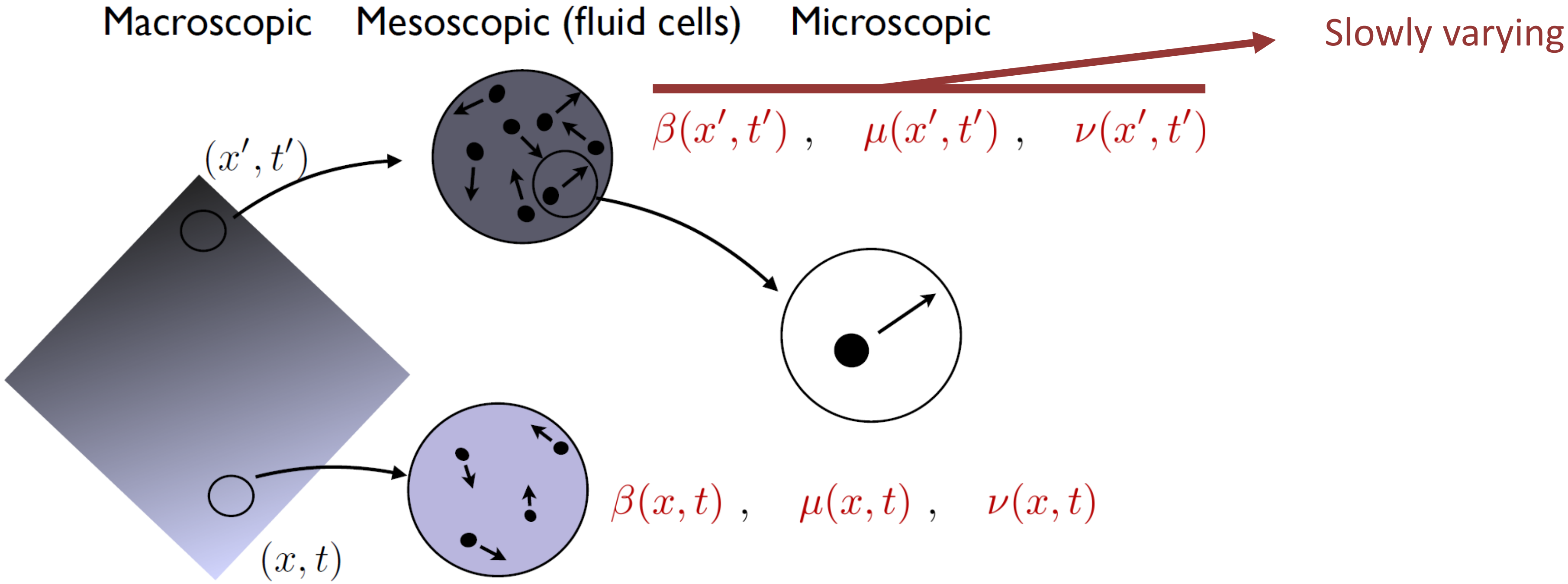

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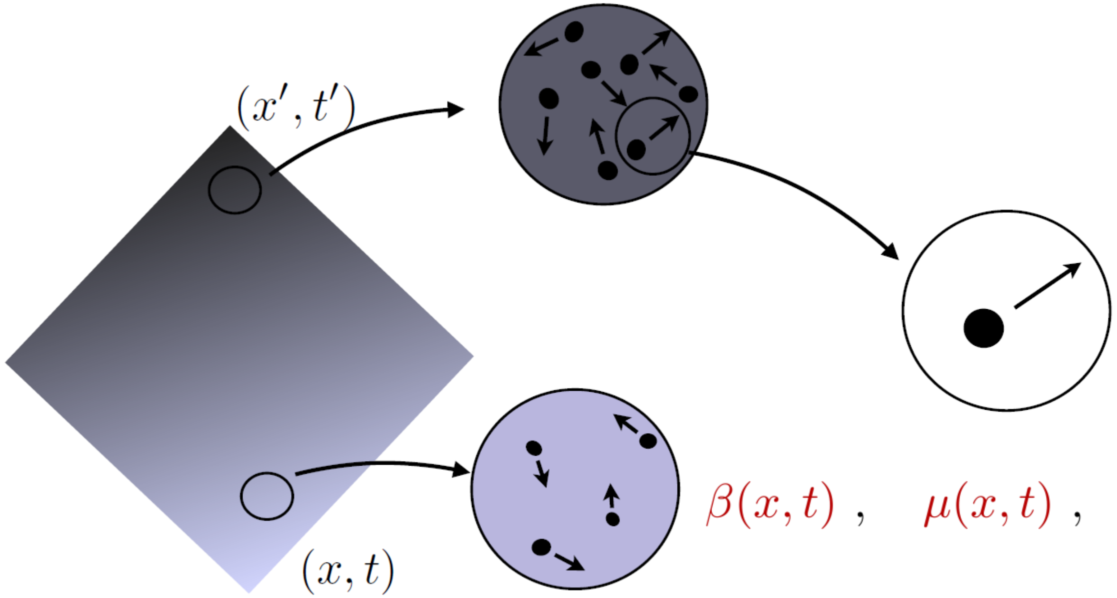
Macroscopic Mesoscopic (fluid cells) Microscopic



$\beta(x,t)$, $\mu(x,t)$, $\nu(x,t)$

$\Rightarrow$ Local GE averages

$\bar{q}_m(x,t)$ and $\bar{j}_m(x,t)$

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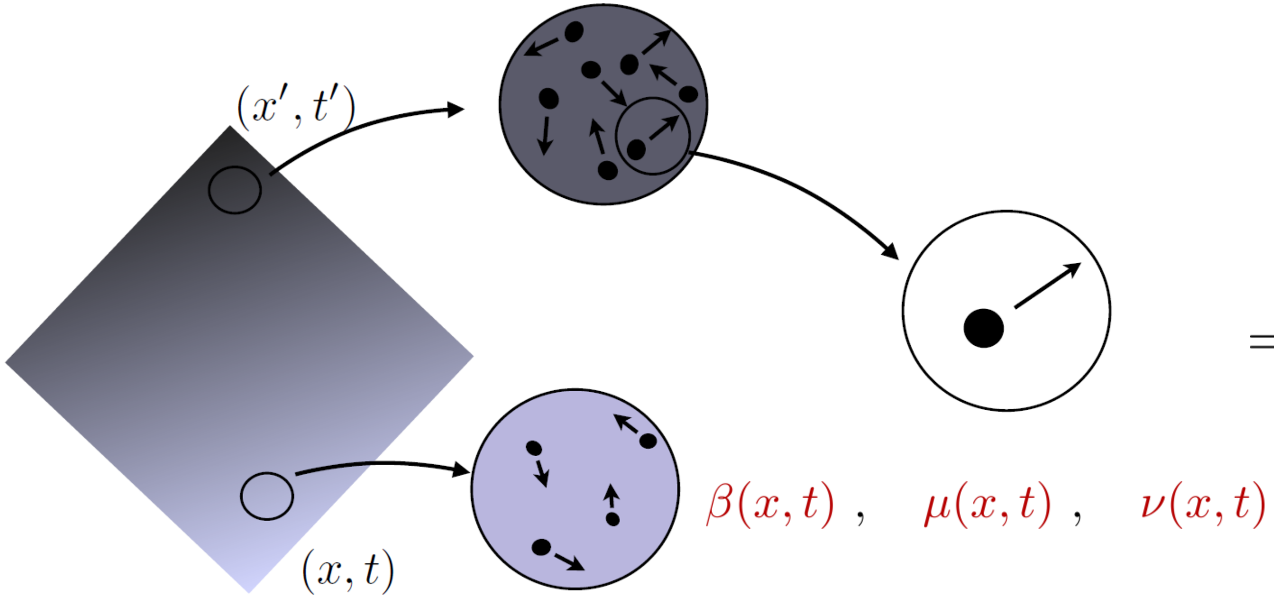

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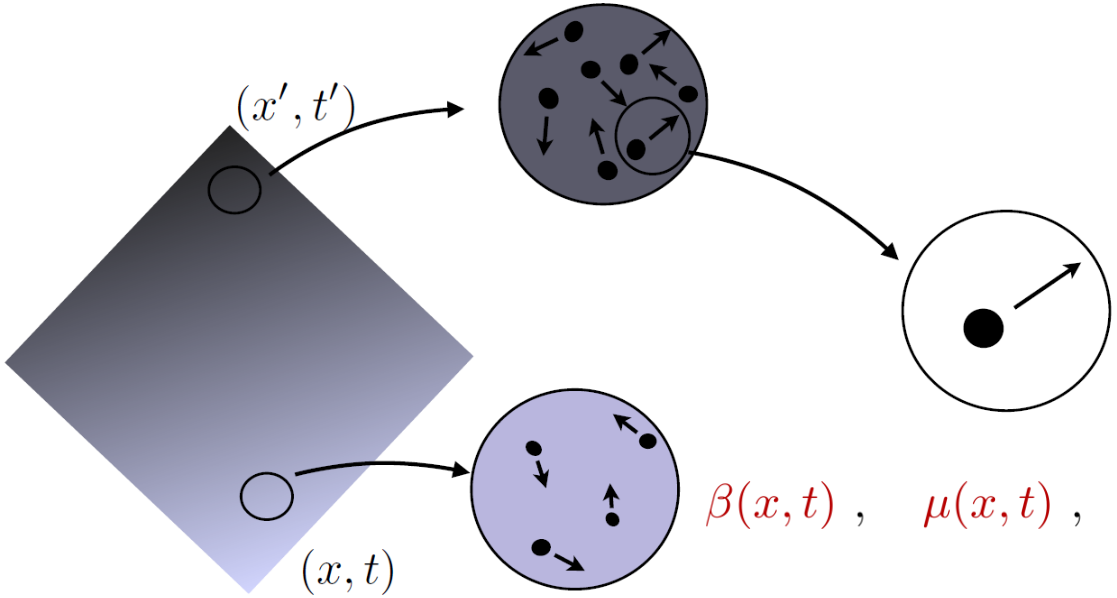

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Functions of $\{\bar{q}_n\}$'s !

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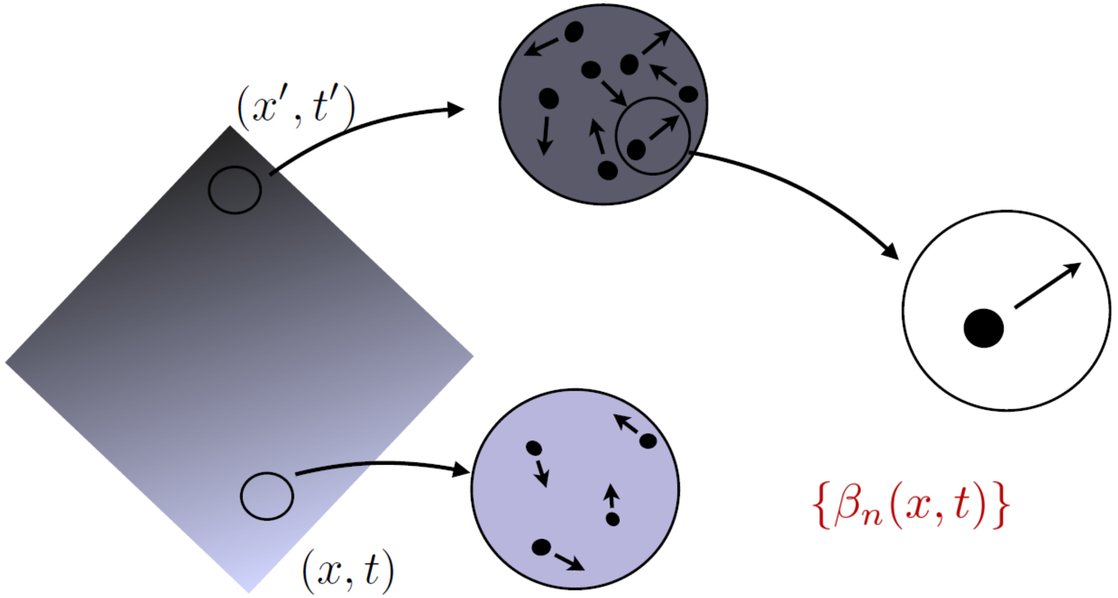

$\{\beta_n(x,t)\}$

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GHD from scattering theory: an example

- KdV: integrable nonlinear dispersive PDE

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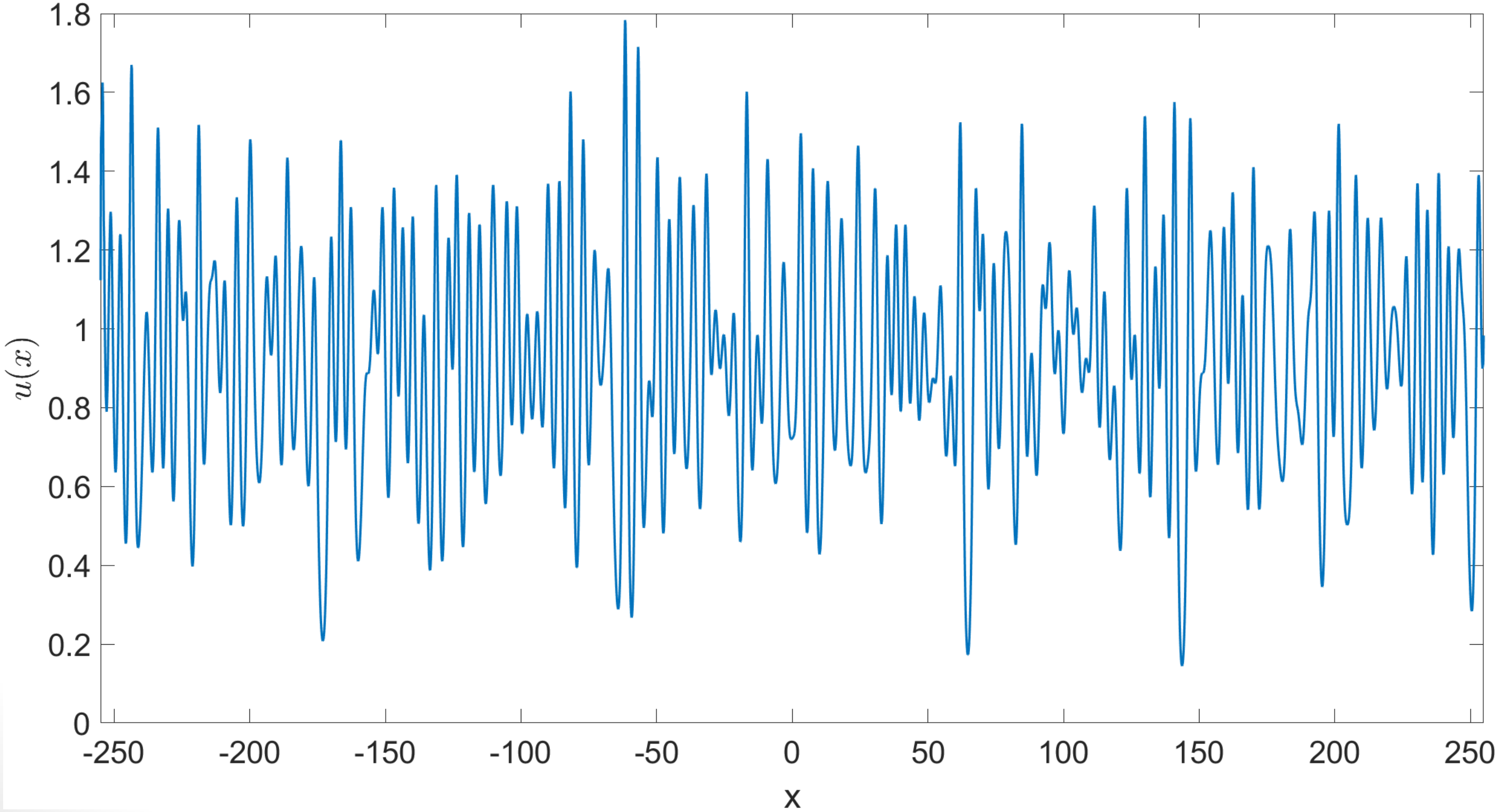

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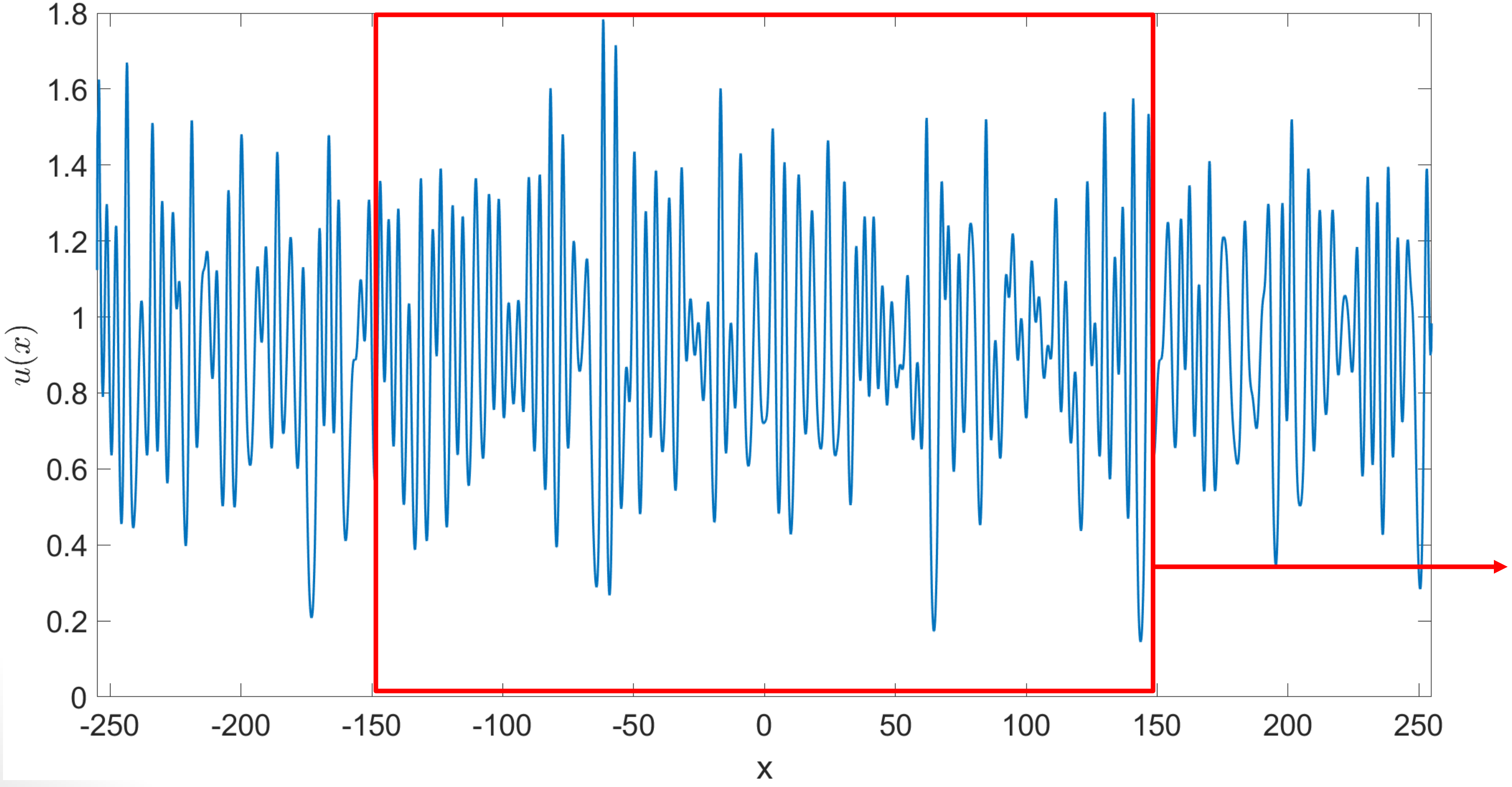

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Fluid cell of size L characterised by local GGE

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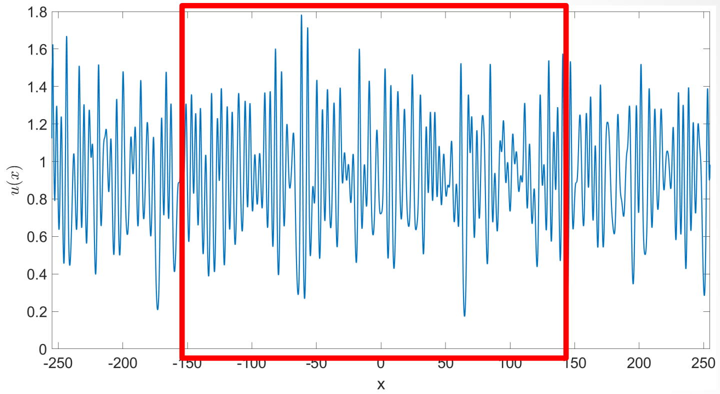
N solitons



- Multisoliton solution

$$u_N \sim \sum_{i=1}^{N} 2\eta_i^2 \mathrm{sech}^2 \left[\eta_i \left(x - 4\eta_i^2 t - x_i^{\pm}\right)\right] \quad \text{as} \quad t \to \pm\infty.$$

[Zakharov (1971)]

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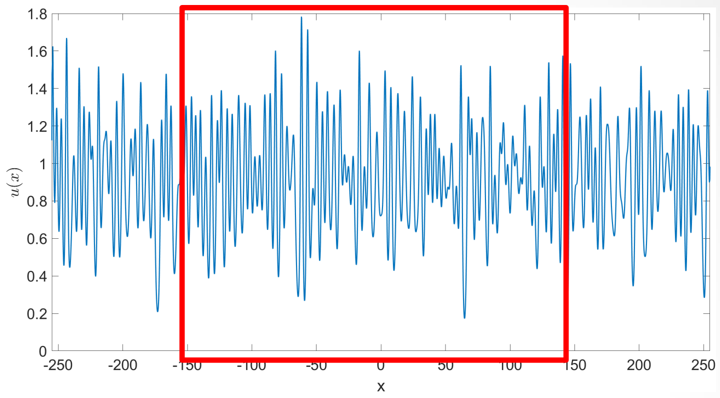
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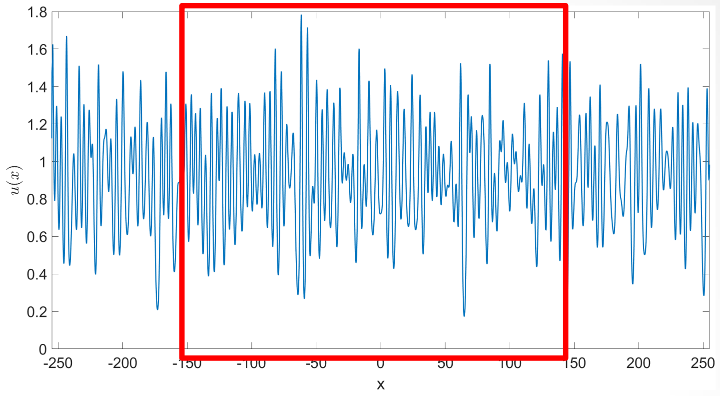
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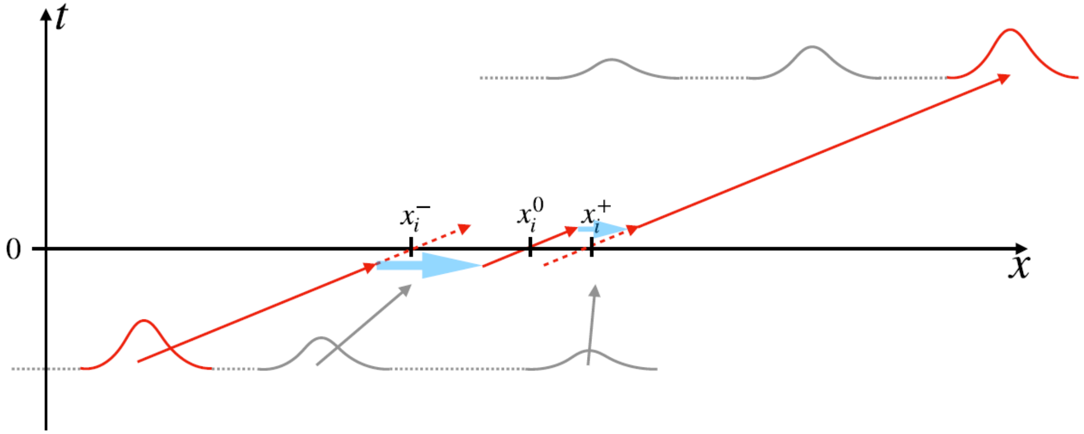
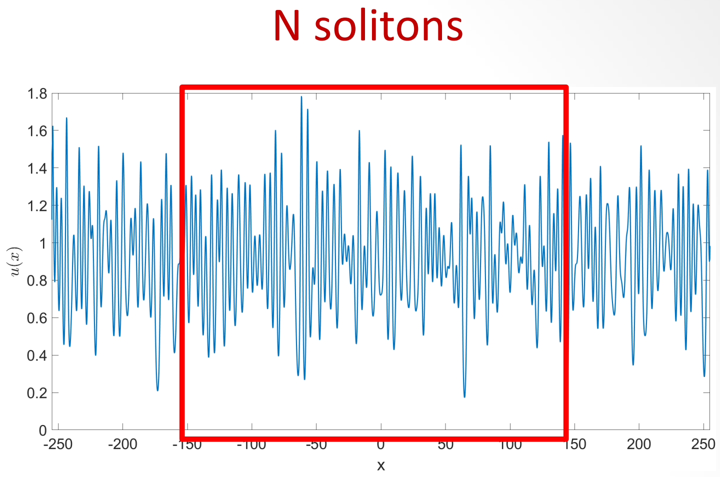


Scattering is elastic and
2-body factorisable

GHD from scattering theory: an example



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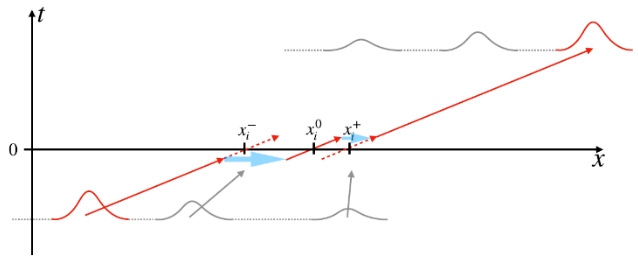
- Relation between asymptotic states given by scattering shift



$$x_i^+ - x_i^- = \sum_j \frac{\operatorname{sgn}(\eta_\mathrm{i} - \eta_\mathrm{j})}{\eta_i} \ln \left| \frac{\eta_i + \eta_j}{\eta_i - \eta_j} \right|$$

[Lax (1968)]

Thermodynamics

- Partition function

$$\mathcal{Z} = \sum_{N=0}^{\infty} \frac{1}{N!} \int_{\Gamma^N \times \mathbb{R}^N} \prod_{i=1}^{N} \frac{\mathrm{d}p(\eta_i)}{2\pi} \mathrm{d}x_i^- \exp\left[-\sum_{i=1}^{N} w(\eta_i)\right] \chi\left(u_N(x, t=0) < \epsilon_x, x \notin [0, L]\right)$$

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Spectral scaling function

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→ Spectral scaling function

- NDR of soliton gases

$$\sigma(\eta)\rho(\eta) = \eta - \int_\Gamma \mathrm{d}\mu \ \rho(\mu) \log\left|\frac{\eta+\mu}{\eta-\mu}\right|$$

$\rho(\eta)\mathrm{d}\eta\mathrm{d}x = \#$ of solitons in $[x, x+\mathrm{d}x] \times [\eta, \eta+\mathrm{d}\eta]$

Density of States

[El (2003)]

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Density of States

[El (2003)]

- Alternative interpretation

$$\frac{\mathrm{d}x^-(\eta)}{\mathrm{d}x} = \frac{\sigma(\eta)\rho(\eta)}{\eta}$$

Change of metric

From thermodynamics to hydrodynamics

- Integrability: infinite number of conservation laws

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$$\Big\Downarrow$$

Fluid cell average

$$\partial_t \bar{q}_n(x,t) + \partial_x \bar{j}_n(x,t) = 0$$

From thermodynamics to hydrodynamics

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A type of (2+1)d GHD featuring line solitons

- Inspired by the phenomenology of the KP equation

$$\partial_x(\partial_t u + 6u\partial_x u + \partial_{xxx} u) \pm 3\partial_{yy} u = 0$$

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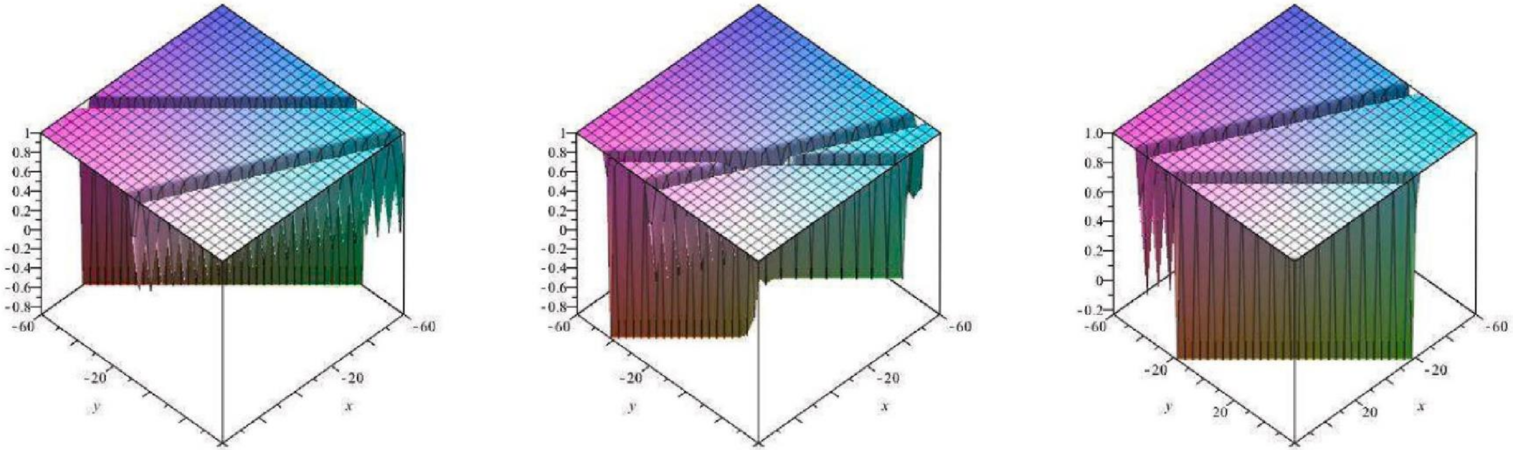
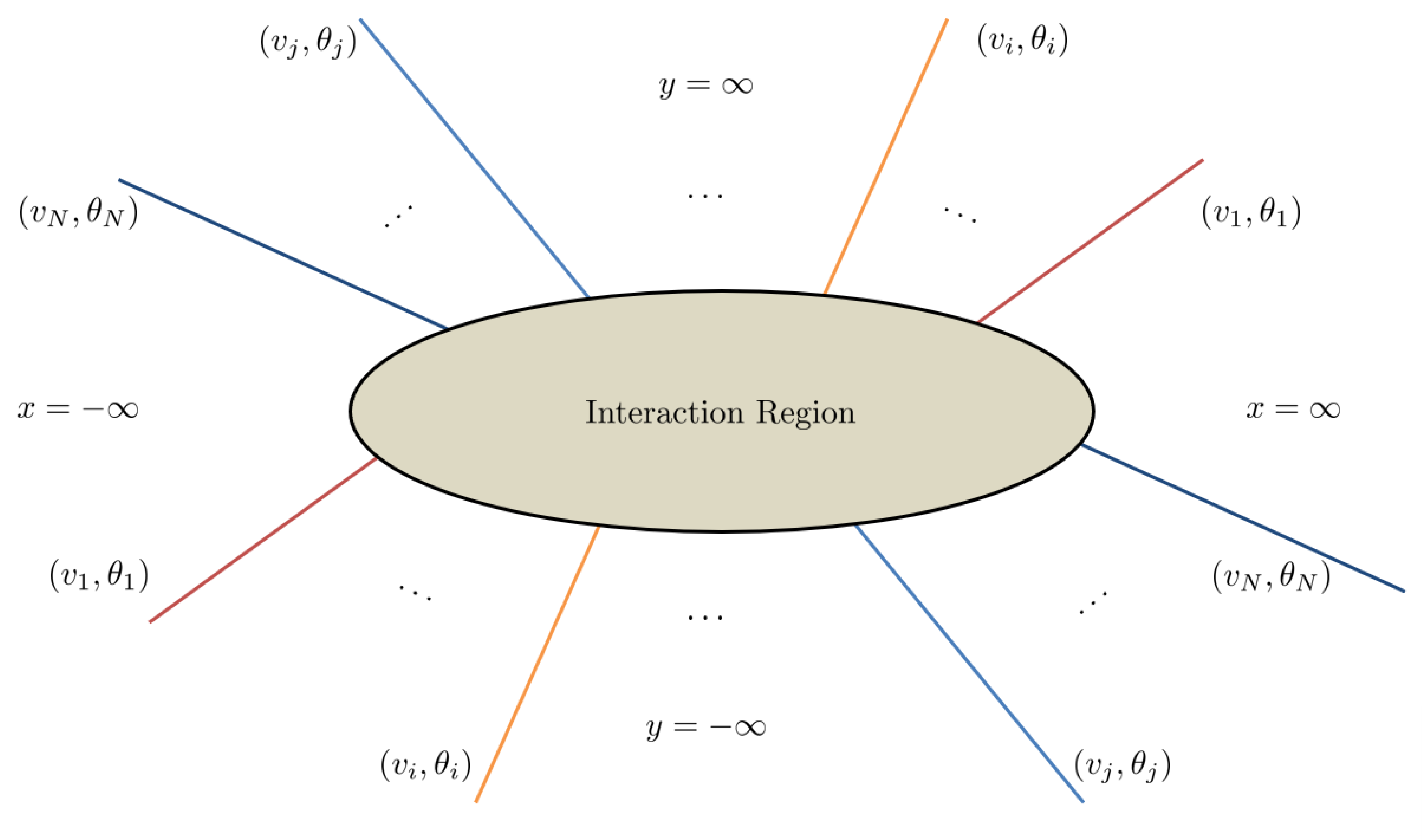
- Elastic and factorised scattering



(a), $t = -5.$ (b), $t = 0.$ (c), $t = 5.$

[Yang (2023)]

Gas of « lines »


(v_j, θ_j)
(v_i, θ_i)
$y = \infty$
(v_N, θ_N)
$\cdots$
(v_1, θ_1)
$x = -\infty$
Interaction Region
$x = \infty$
(v_1, θ_1)
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$\cdots$
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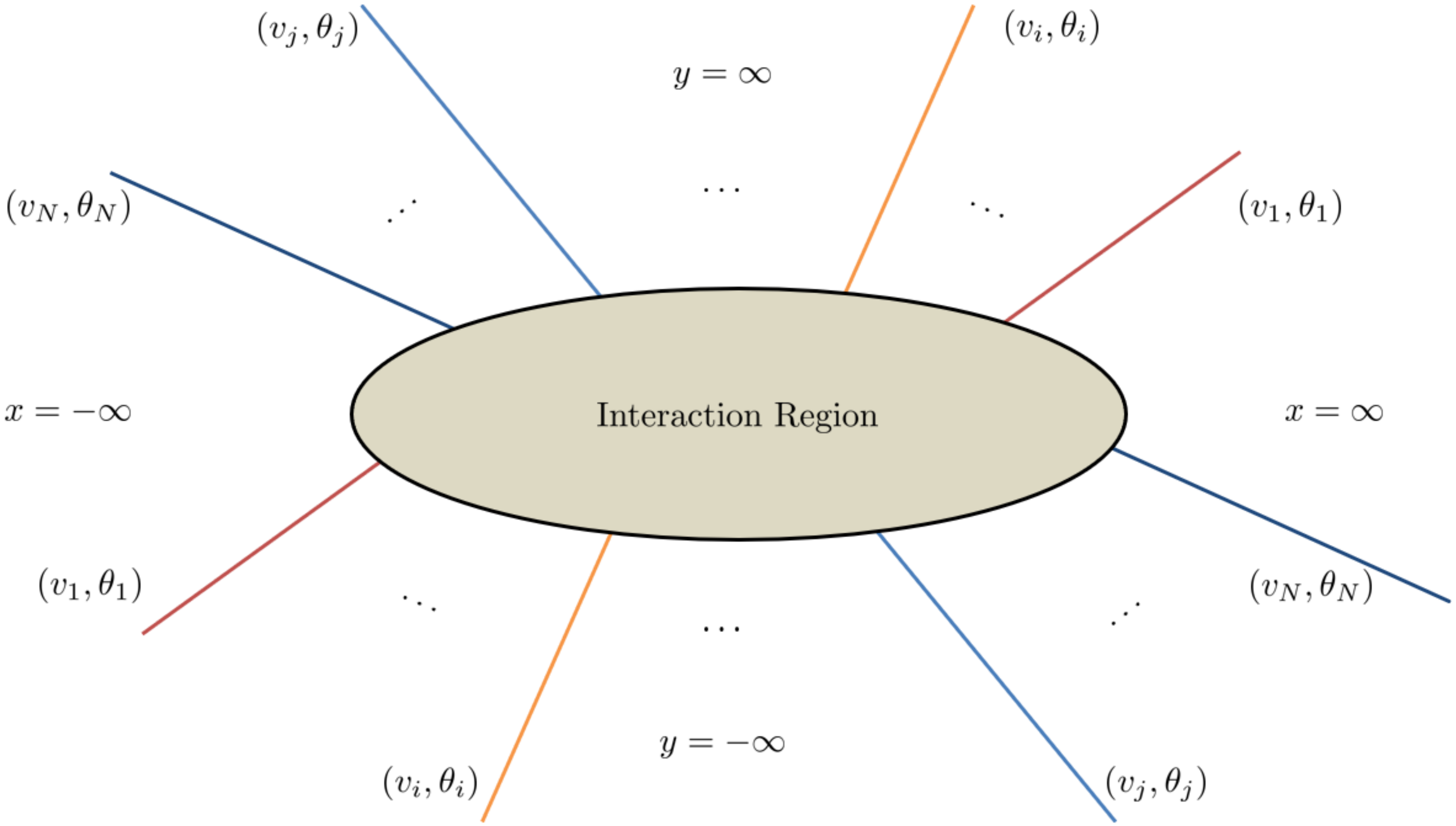
(v_j, θ_j)

Gas of « lines »



Assumptions

- $\theta_i \neq \theta_j$, $i \neq j$

$\Rightarrow$ IR is finite
$\Rightarrow$ Every soliton interacts with every other in the IR

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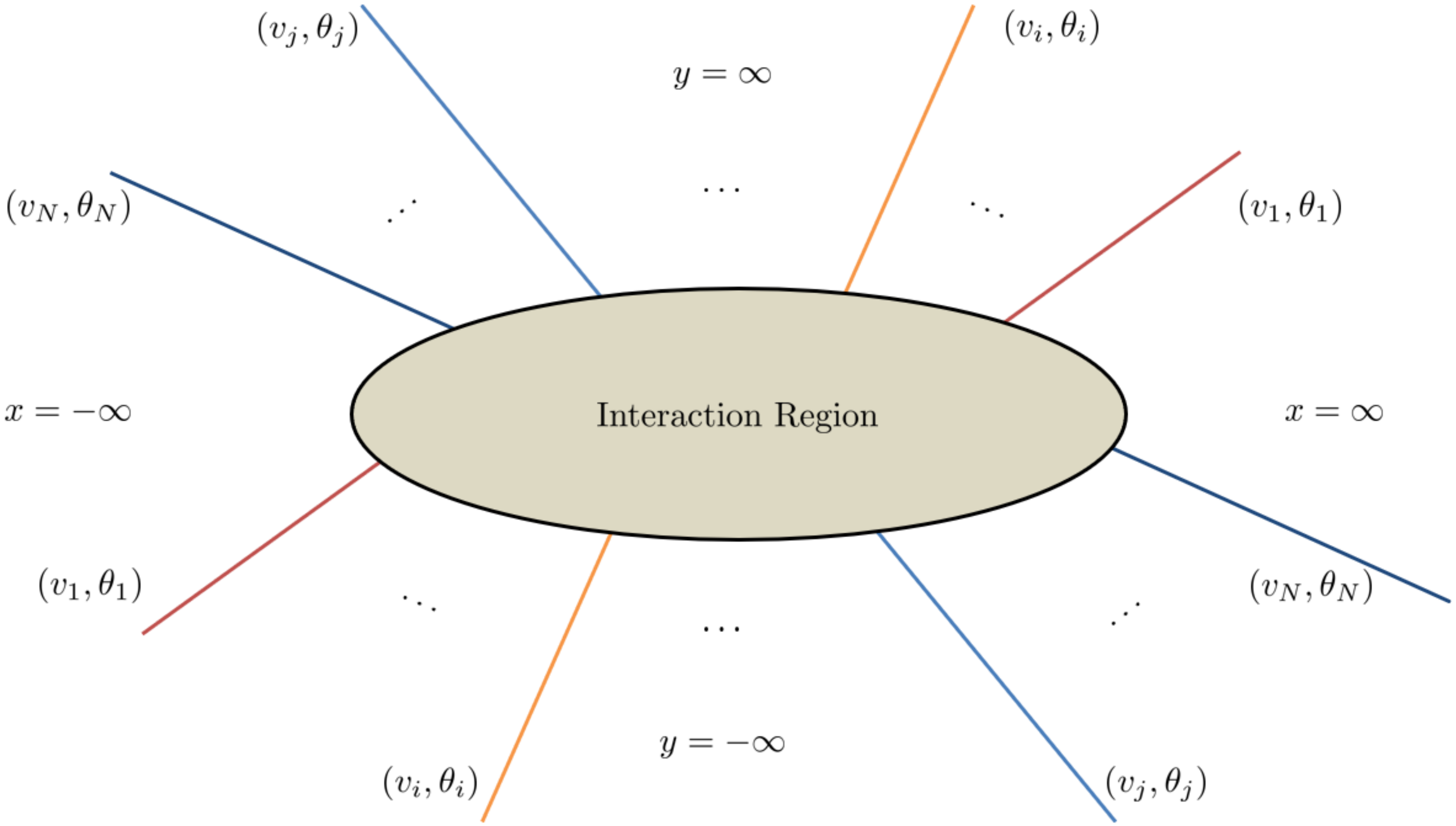

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$$K_{ij} \equiv K(v_i, \theta_i; v_j, \theta_j)$$

$$K_i = \sum_{j \neq i} K_{ij}$$

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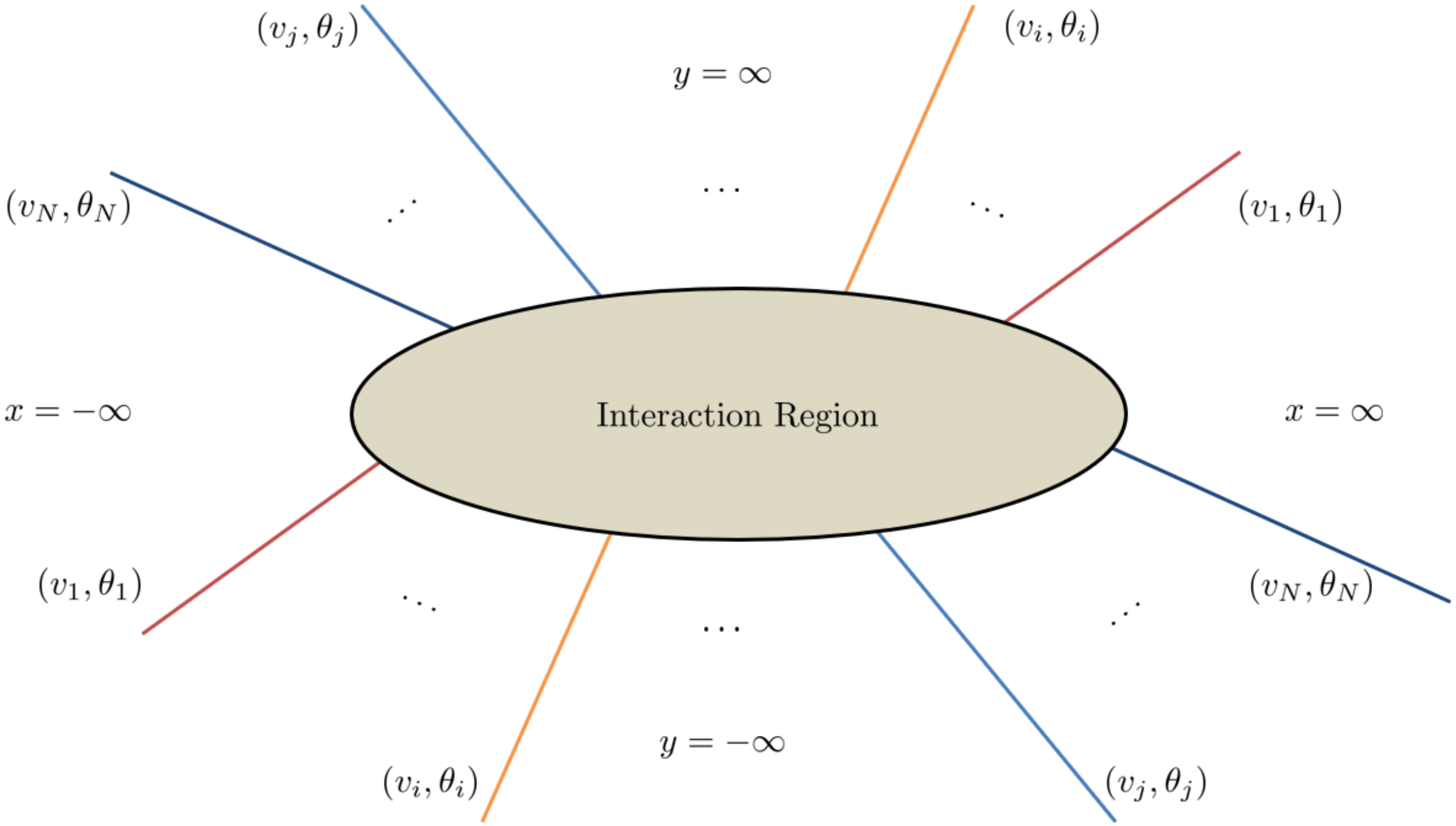

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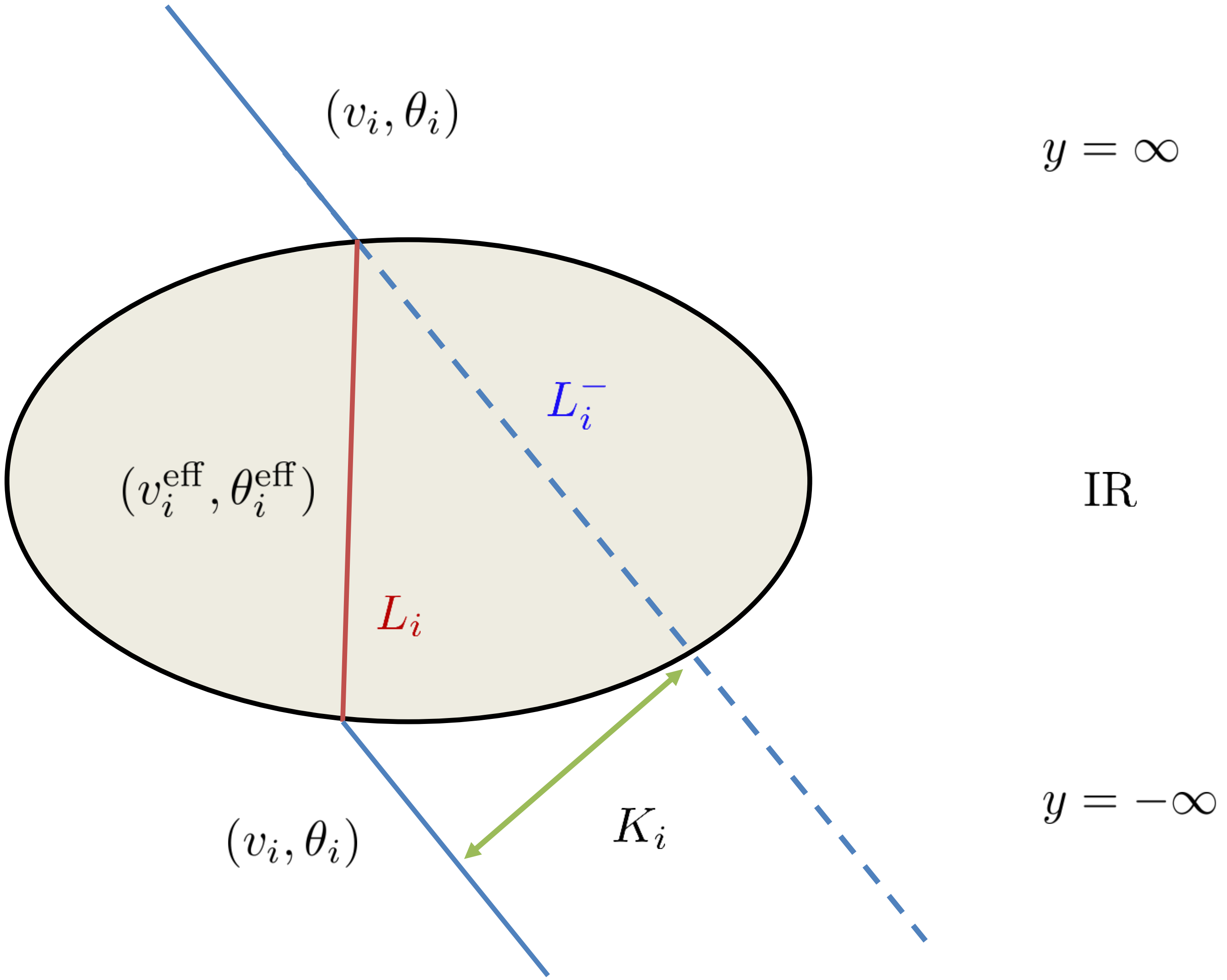
- Homogeneous gas in bulk of IR

Effective orientation



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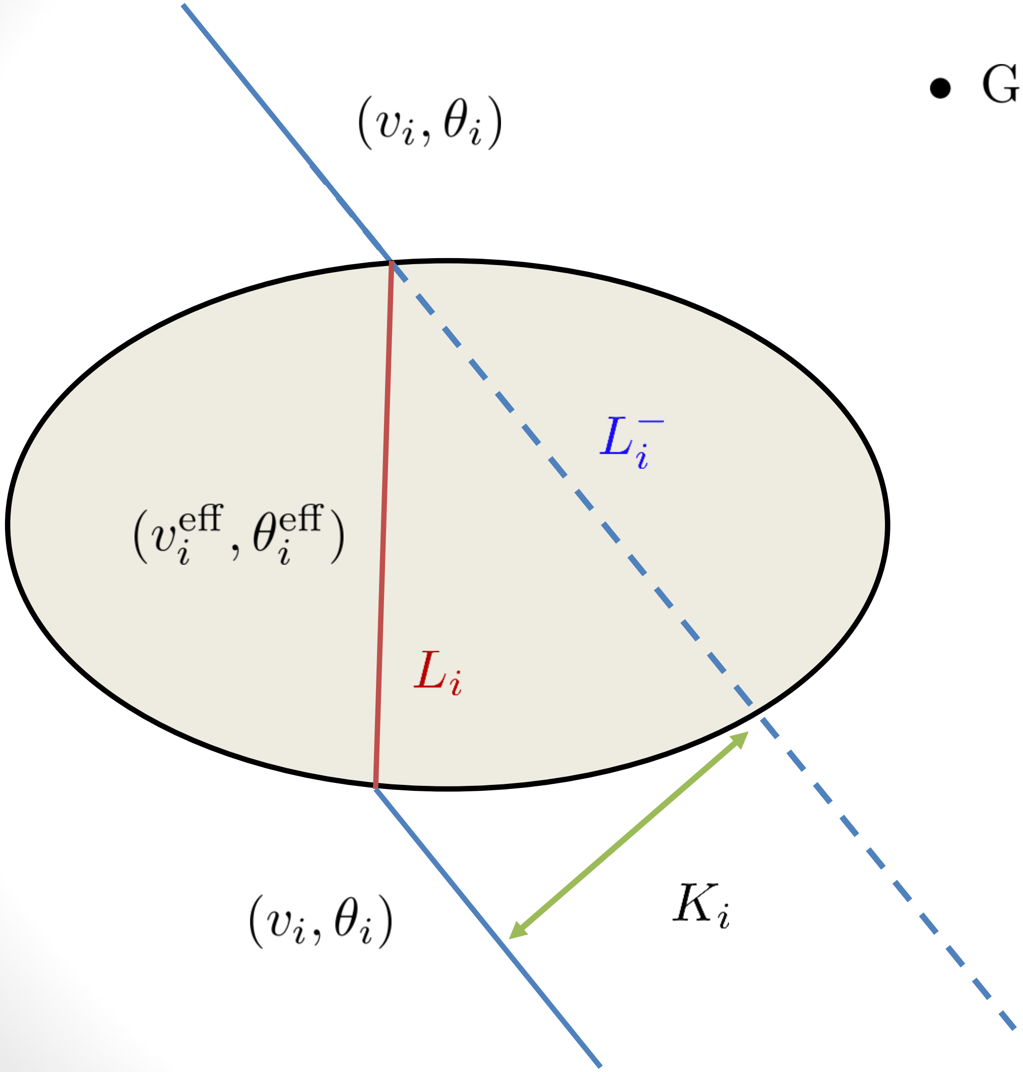

- Geometric argument

$$\theta_i^{\text{eff}} = \theta_i - \arcsin \frac{K_i}{L_i}$$

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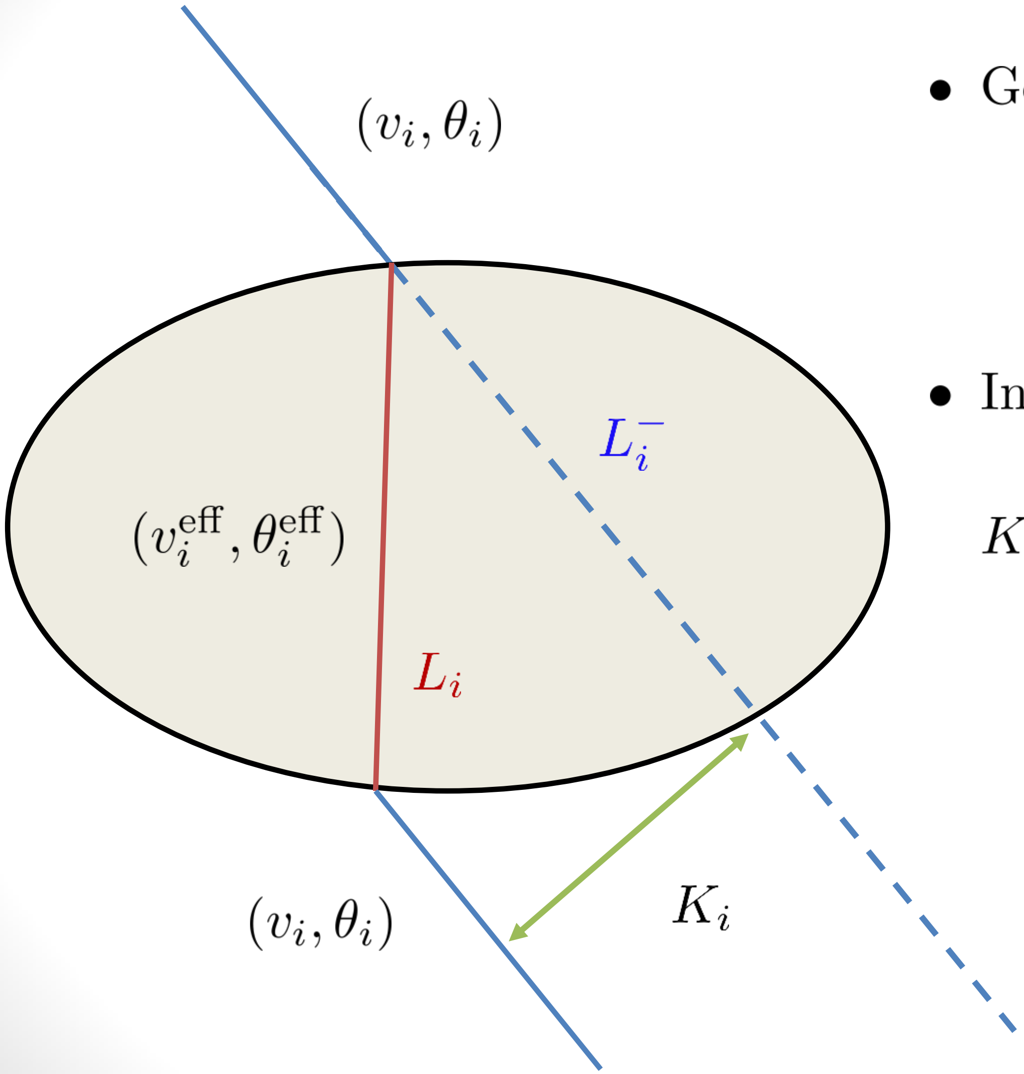

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$$K_i = \sum_j K_{ij} \approx L_i \int \tilde{\rho}_i(u, \alpha) K(v_i, \theta_i; u, \alpha) \mathrm{d}u \mathrm{d}\alpha$$

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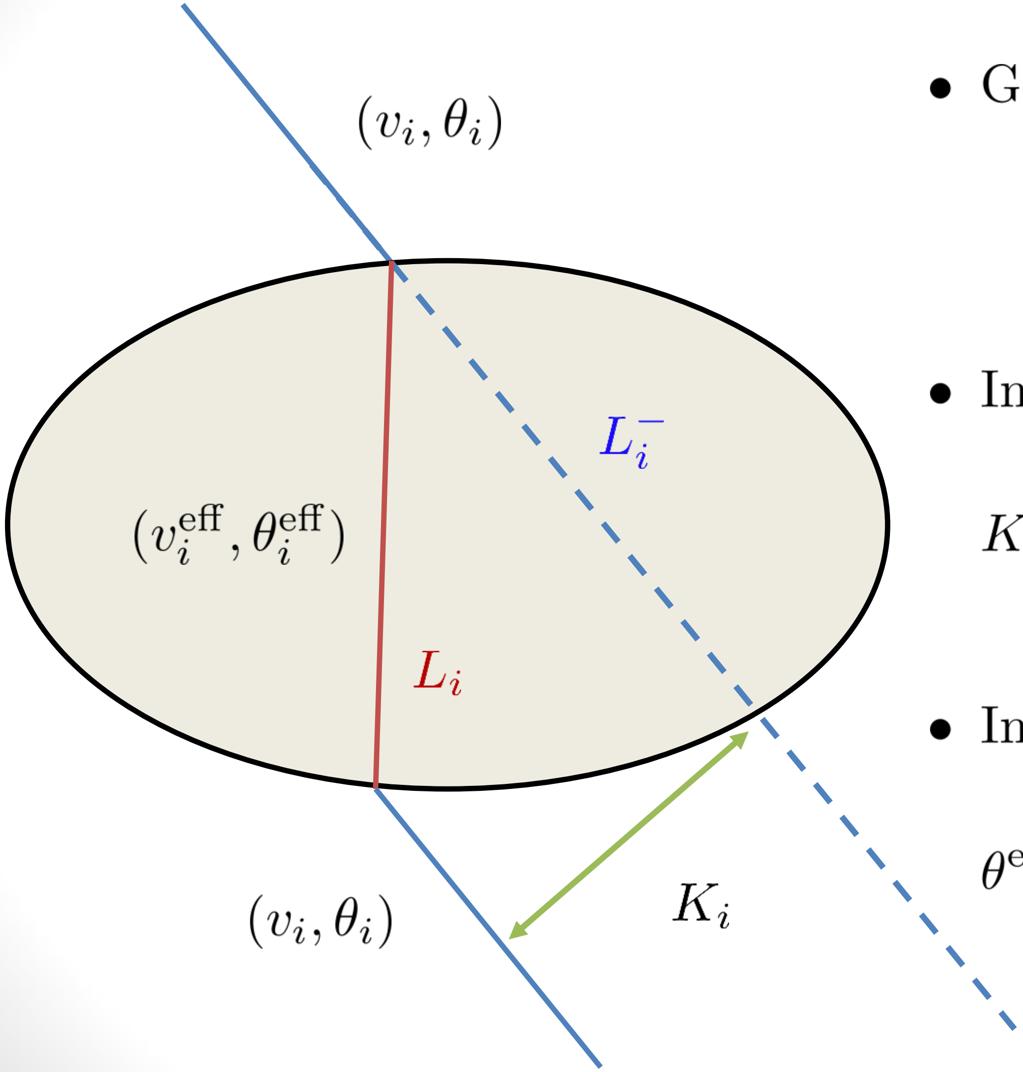

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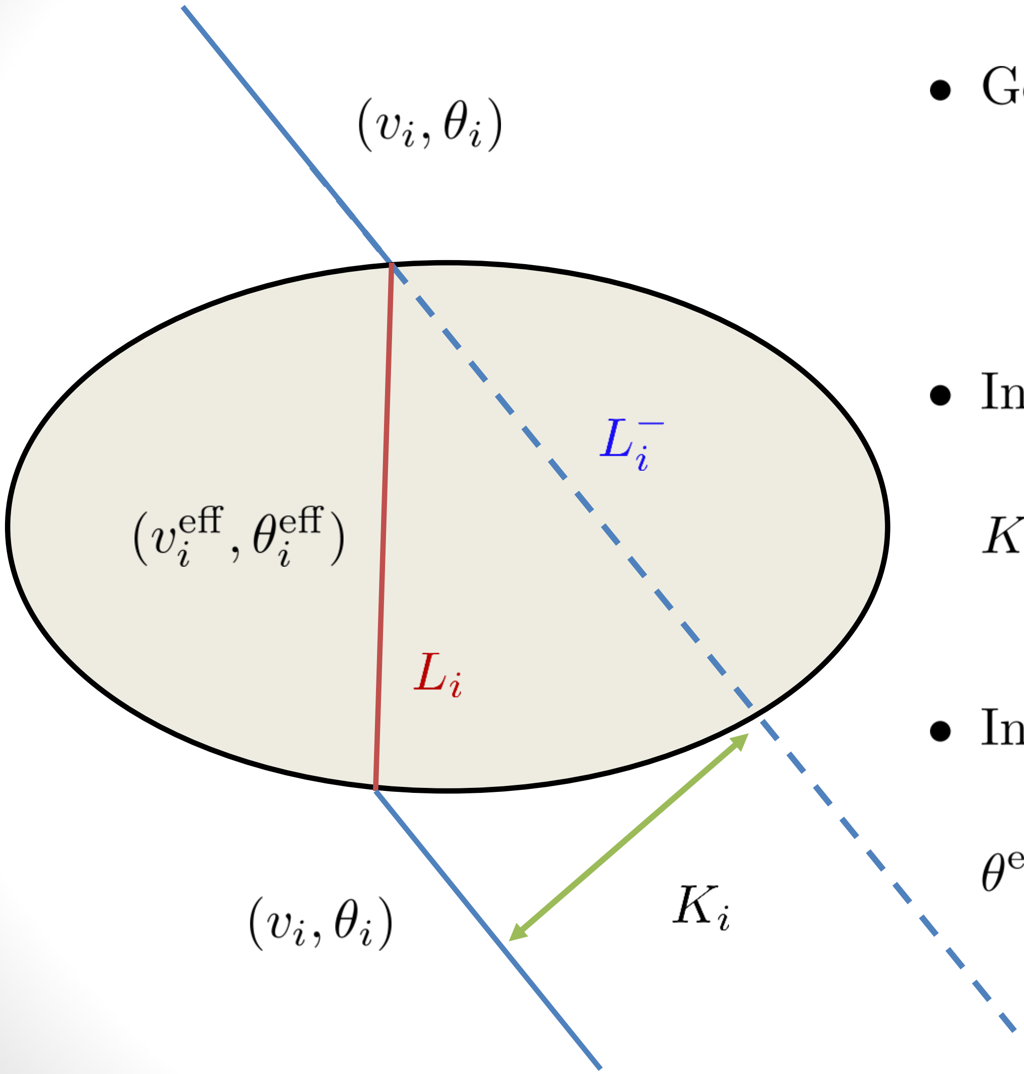
$$\theta^{\text{eff}}(v, \theta) = \theta - \arcsin \left[\int \tilde{\rho}(u, \alpha; \theta^{\text{eff}}(v, \theta)) K(v, \theta; u, \alpha) \mathrm{d}u \mathrm{d}\alpha \right]$$

Effective orientation



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$$\tilde{\rho}(v, \theta; \theta_i^{\text{eff}}) = \tilde{\rho}_i(v, \theta)$$

Line density of reference

- Want to relate $\tilde{\rho}(v, \theta; \varphi)$ to $\tilde{\rho}(v, \theta; 0)$

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- Want to relate $\tilde{\rho}(v,\theta;\varphi)$ to $\tilde{\rho}(v,\theta;0)$

$$\tilde{\rho}(v,\theta;0) \sim \frac{1}{\underline{\Delta(v,\theta)}} \qquad \text{and} \qquad \tilde{\rho}(v,\theta;\varphi) \sim \frac{1}{\Delta'(v,\theta)}$$

Typical distance between two solitons with parameters in the vicinity of (v,θ) as they intersect the horizontal

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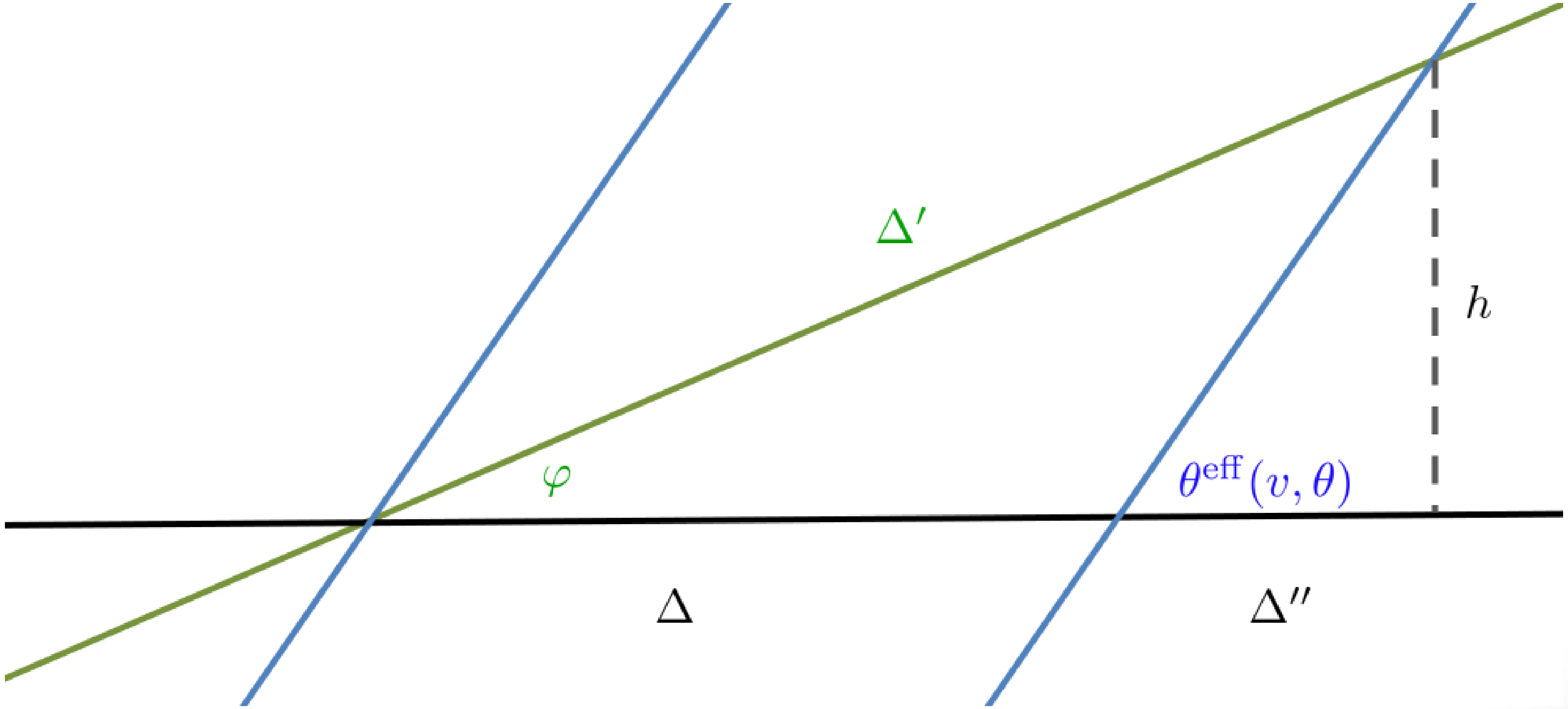

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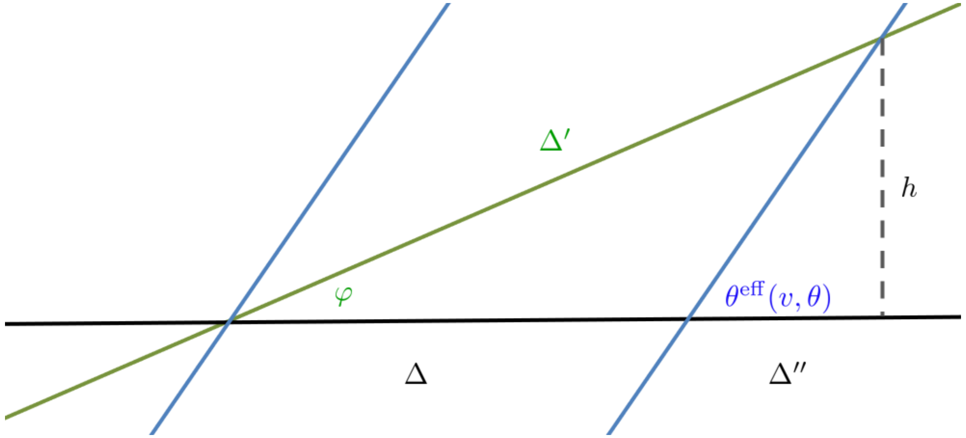

$$\Delta = \Delta' \sin\varphi \left(\cot\varphi - \cot\theta^{\mathrm{eff}}\right)$$

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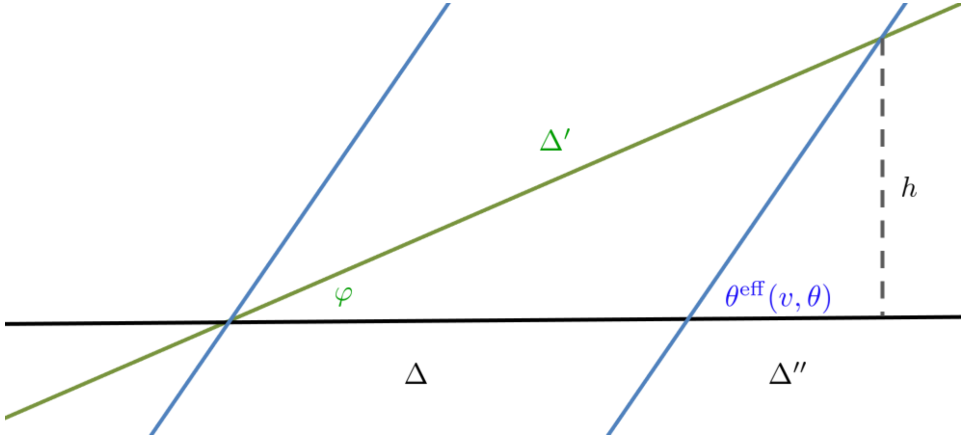

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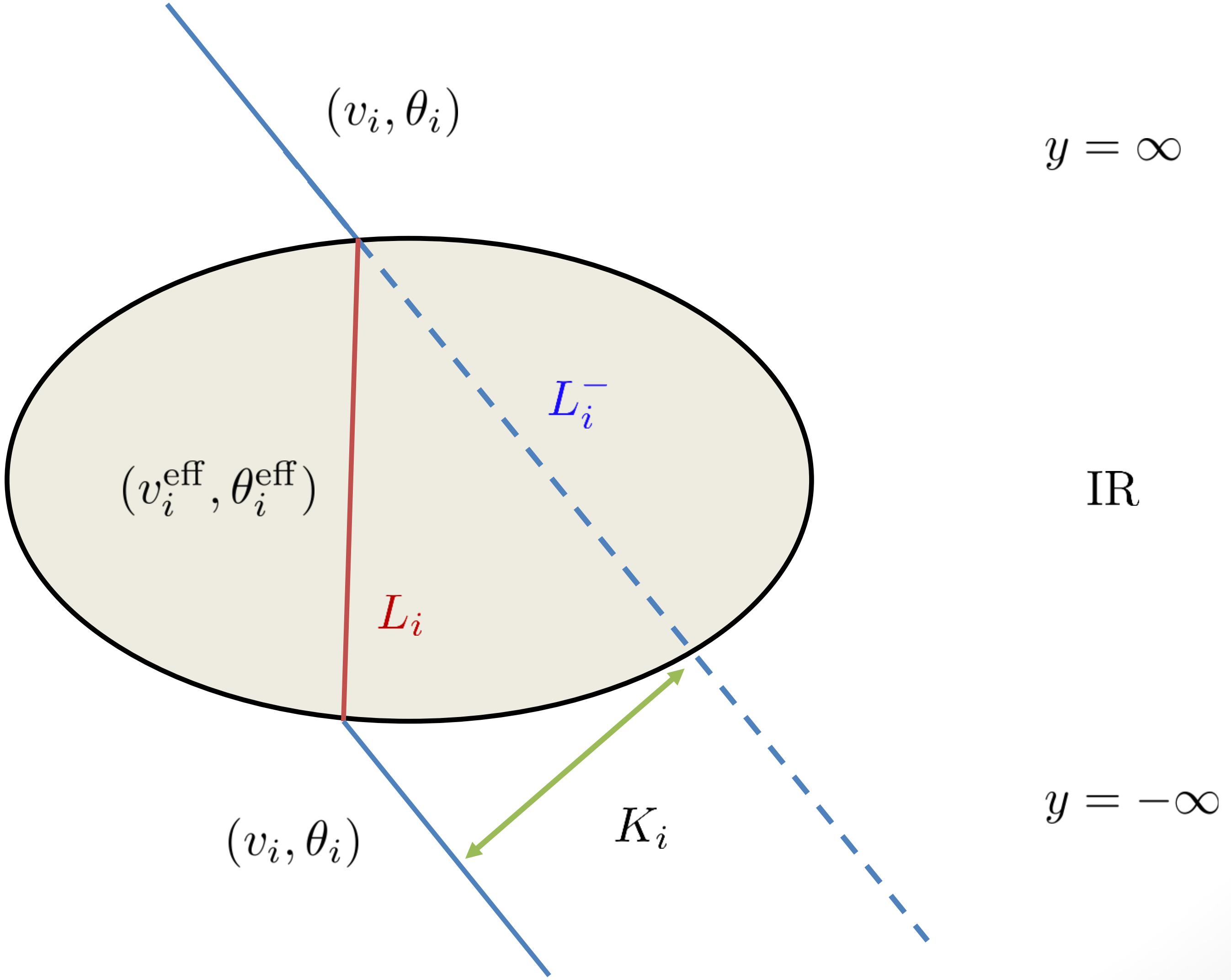
$$\theta^{\text{eff}}(v,\theta) = \theta - \arcsin\left[\int \tilde{\rho}(u,\alpha;0)K(v,\theta;u,\alpha)\left|\sin\theta^{\text{eff}}(v,\theta)\right|\left|\cot\theta^{\text{eff}}(v,\theta) - \cot\alpha^{\text{eff}}(u,\alpha)\right| \mathrm{d}u\mathrm{d}\alpha\right]$$

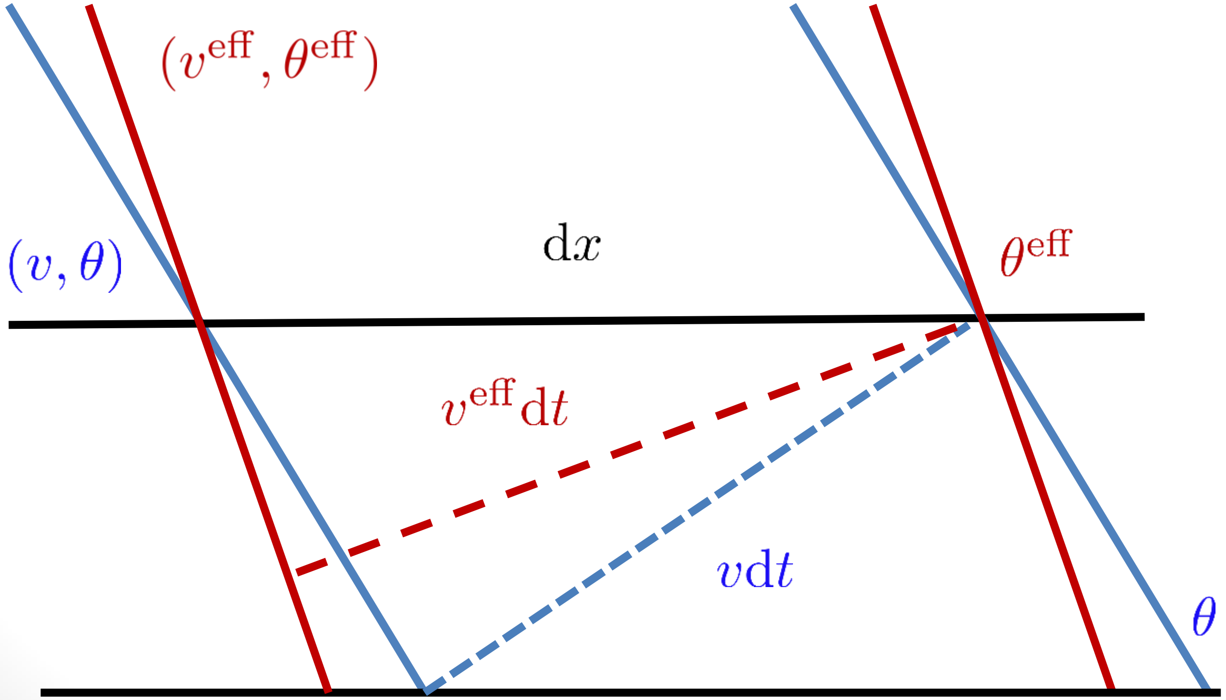
Refraction and effective velocity

- Similarities with refraction: there should be a way to relate $\theta^{\rm eff}$ and $v^{\rm eff}$



Refraction and effective velocity

- Similarities with refraction: there should be a way to relate θ^{eff} and v^{eff}
- Geometric argument: assume that over $\mathrm{d}t$ IR does not change and over $\mathrm{d}x$ it can be considered flat



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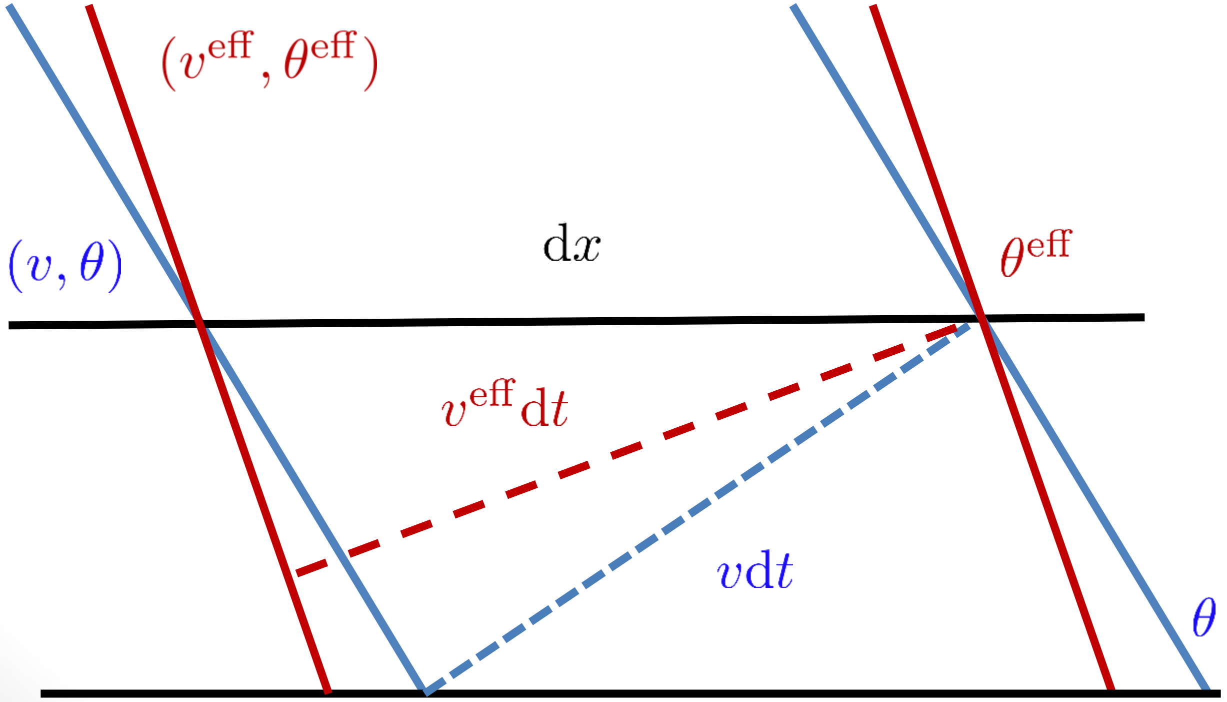

$$\frac{v^{\mathrm{eff}}}{v} = \frac{\sin \theta^{\mathrm{eff}}}{\sin \theta}$$

Snell's law!

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- Remark 1: θ w.r.t the horizontal

Refraction and effective velocity

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Snell's law!

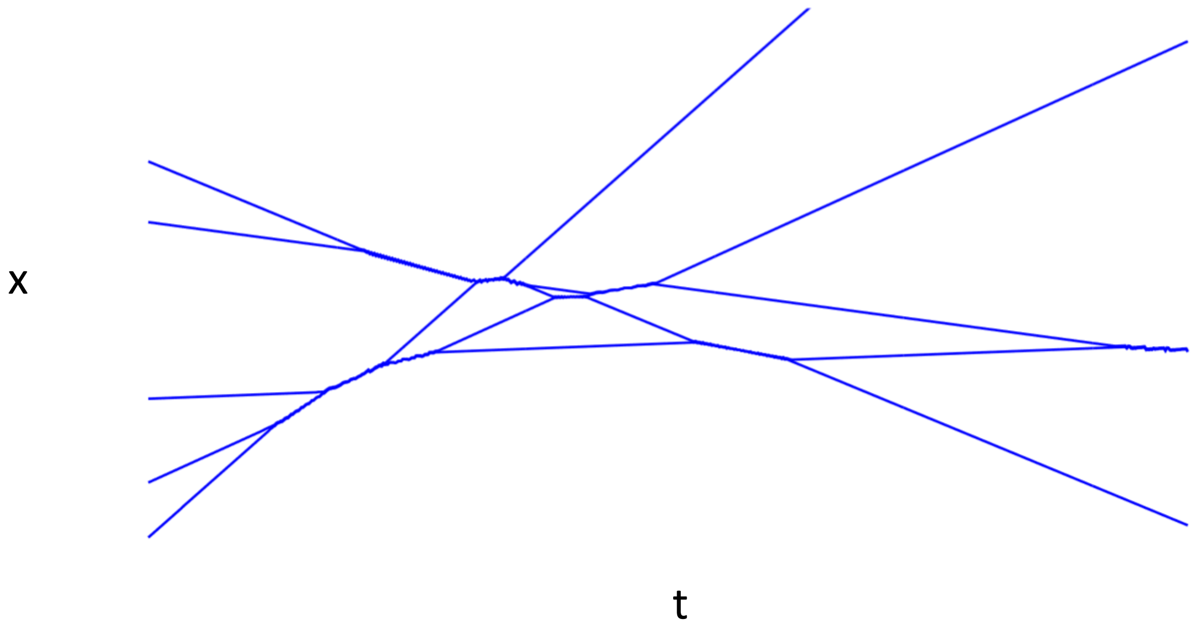
- Remark 1: θ w.r.t the horizontal

- Remark 2: for $\theta^{\text{eff}} \approx \theta$

$$v^{\text{eff}} \approx v\left\{1 - \cot\theta\left[\int \tilde{\rho}(u,\alpha;0)K(v,\theta;u,\alpha)\left|\sin\theta^{\text{eff}}(v,\theta)\right|\left|\cot\theta^{\text{eff}}(v,\theta) - \cot\alpha^{\text{eff}}(u,\alpha)\right|\mathrm{d}u\mathrm{d}\alpha\right]\right\}$$

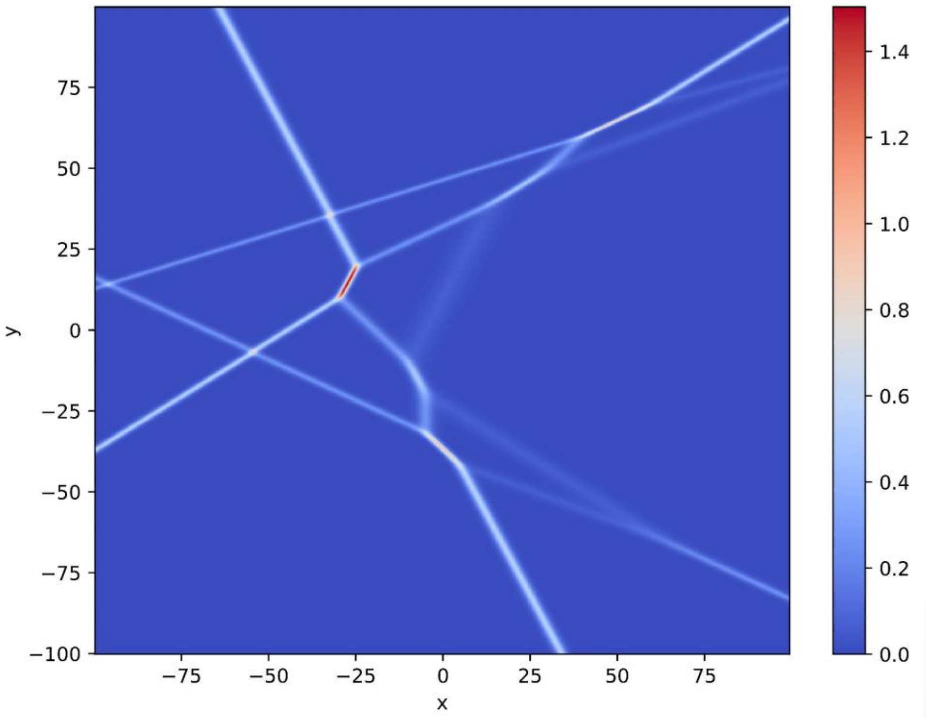
Analogy with (1+1)D models

- Gas of lines allows for analogy with (1+1)D systems



[Courtesy of F. Hubner]

Space-time trajectories of quasi particles in a (1+1)d system



[Courtesy of G. Roberti]

Snapshot of KP N-soliton solution in the (x,y) plane

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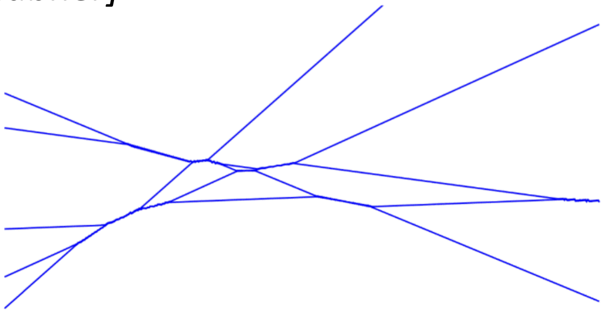

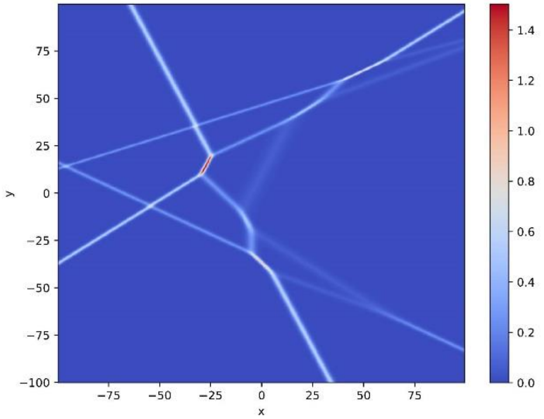

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Snapshot of KP N-soliton solution in the (x,y) plane

- At fixed time KP equation yields an (integrable) Boussinesq equation

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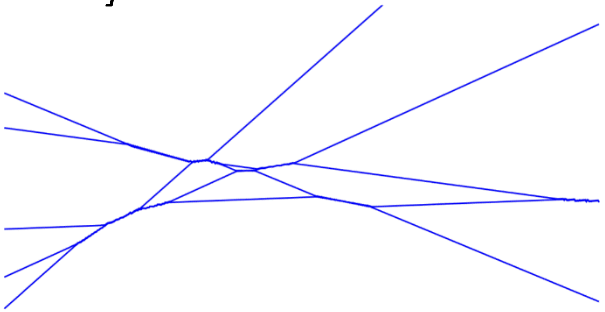

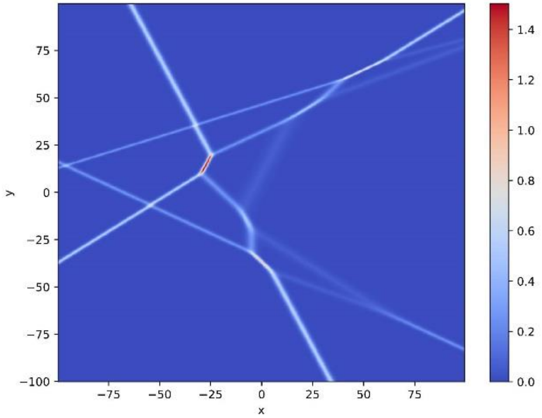

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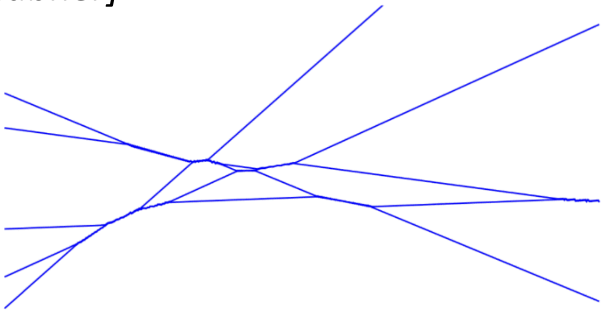

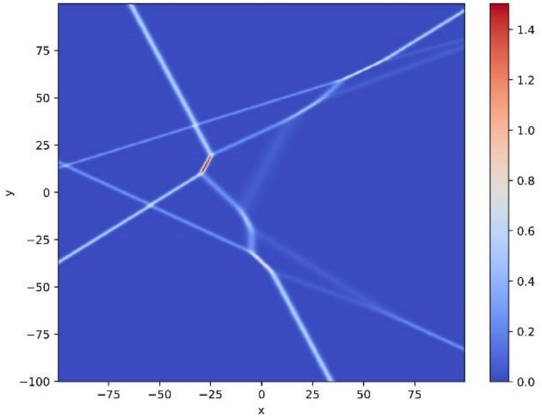

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$\Rightarrow$ Snapshot from (1+1)D space-time trajectories

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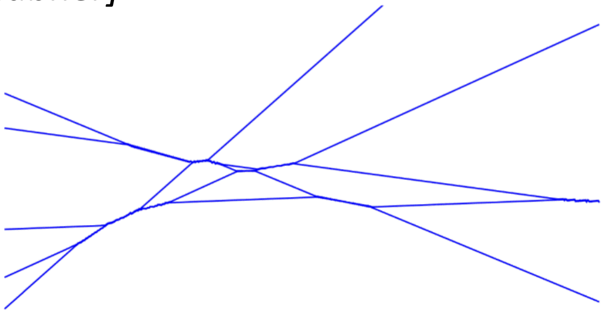
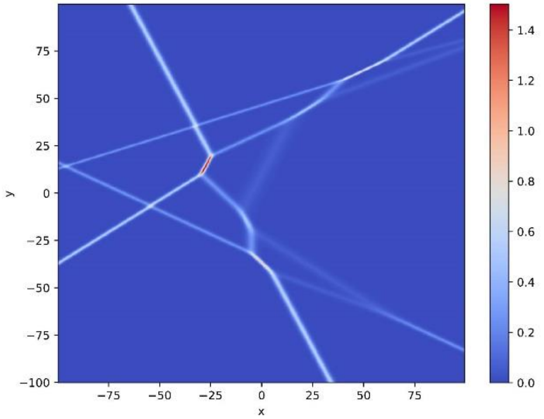
Space-time trajectories of quasi particles in a (1+1)d system





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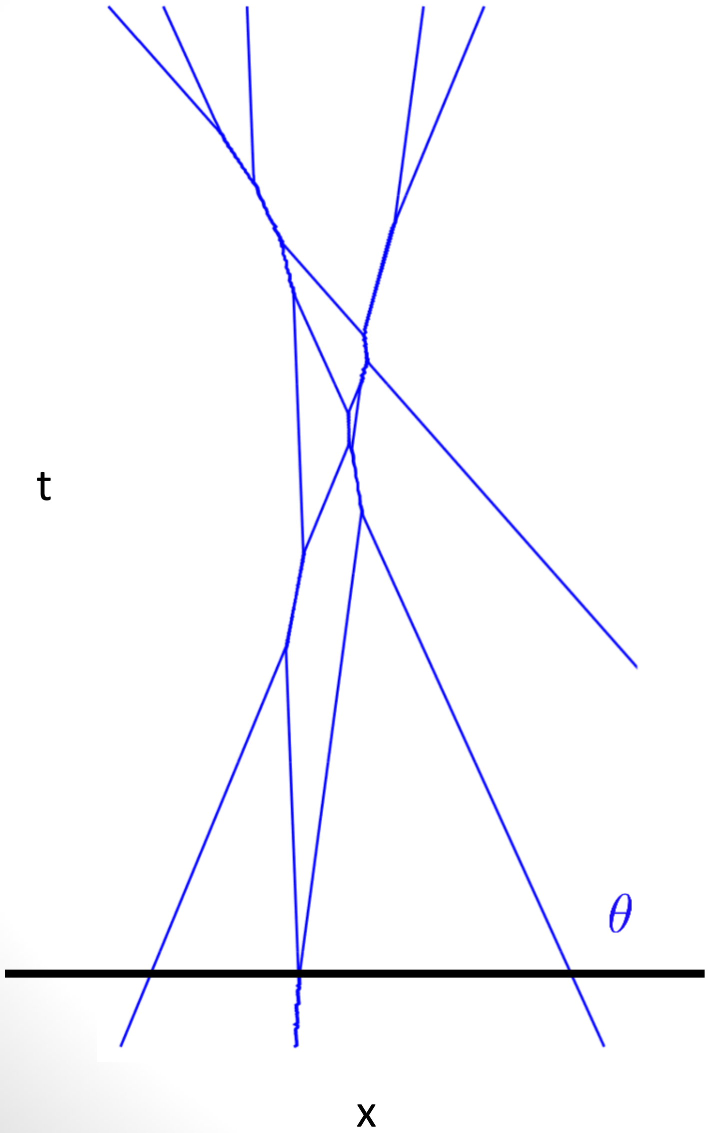
$\Rightarrow$ Dynamics obtained by varying the impact parameters $x^{-}_{1\mathrm{D},i}$ via $v_{2\mathrm{D},i}$

Refraction revisited



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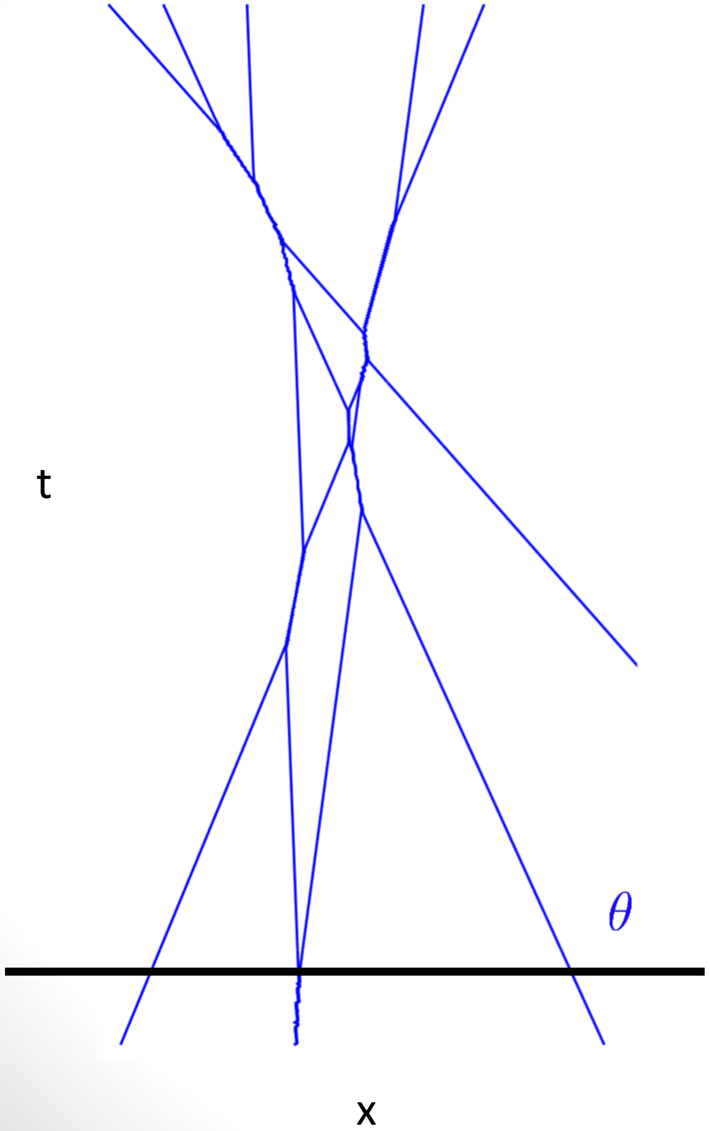
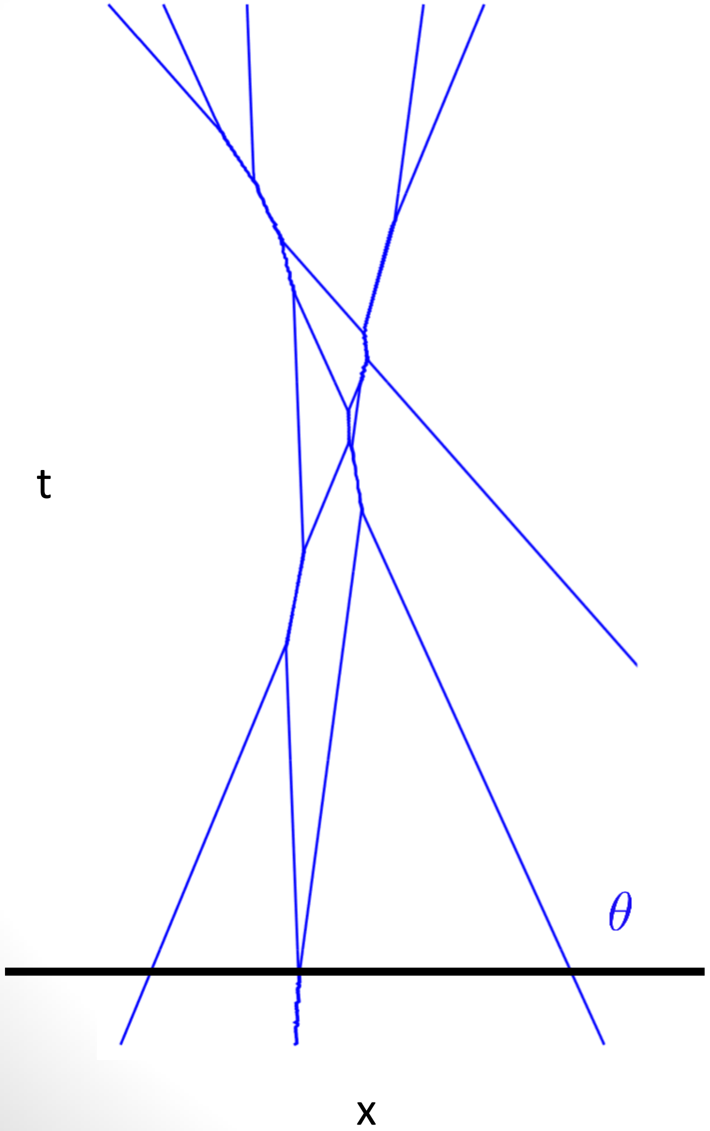


- Identify θ with $v_{1\mathrm{D}}$

$$v_{1\mathrm{D}} \leftrightarrow \frac{1}{\tan\theta} \quad \text{and} \quad v_{1\mathrm{D}}^{\mathrm{eff}} \leftrightarrow \frac{1}{\tan\theta^{\mathrm{eff}}}$$

Refraction revisited



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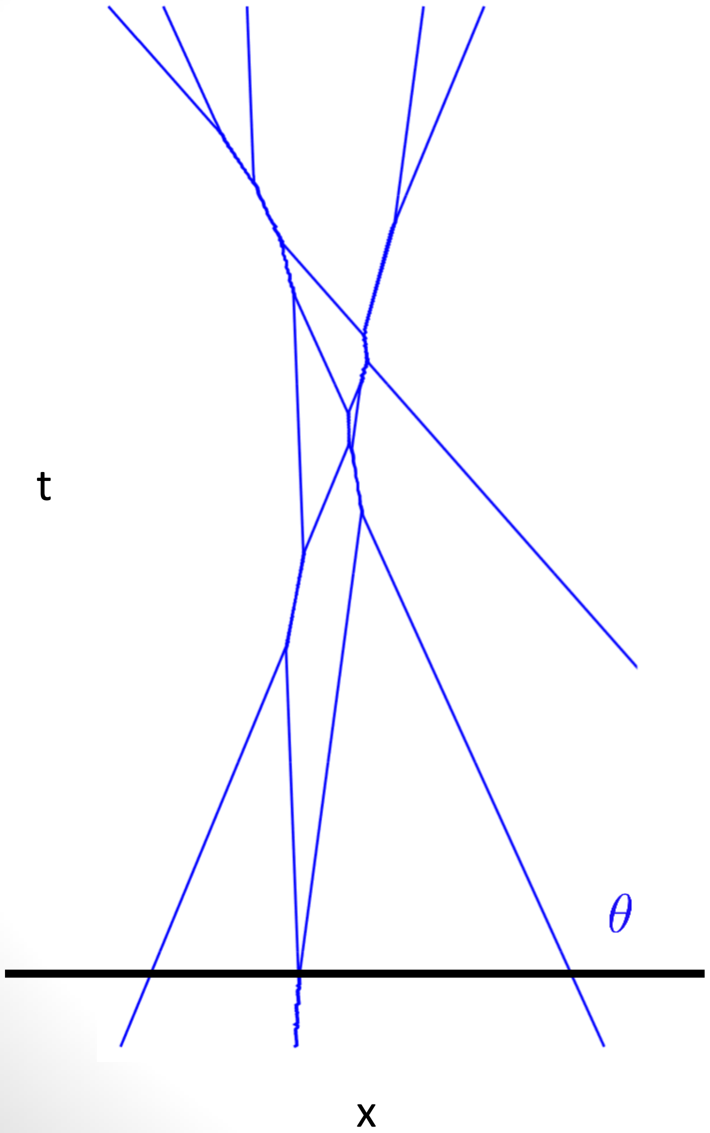
- Recall "Snell's law"

$$\frac{v_{\mathrm{2D}}^{\mathrm{eff}}}{v_{\mathrm{2D}}} = \frac{\sin\theta^{\mathrm{eff}}}{\sin\theta} = \sqrt{\frac{1+\cot^2\theta}{1+\cot^2\theta^{\mathrm{eff}}}}$$

Refraction revisited



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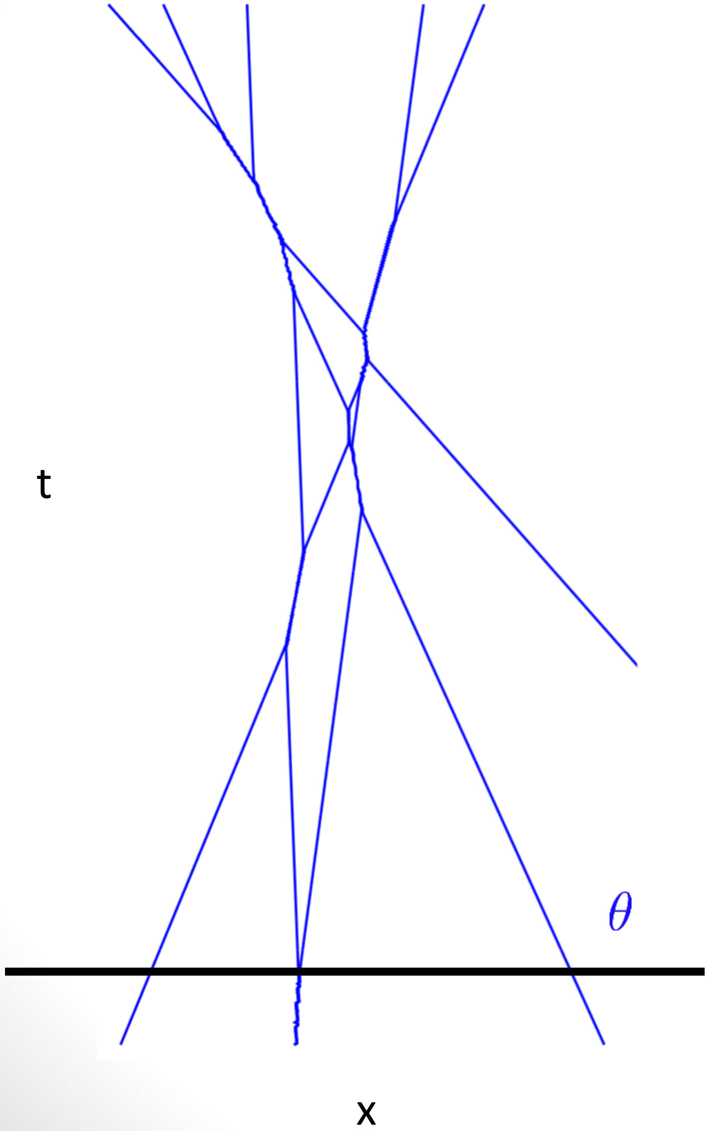
$$\Rightarrow \quad (v_{2\mathrm{D}}^{\mathrm{eff}})^2\left[1+(v_{1\mathrm{D}}^{\mathrm{eff}})^2\right] = v_{2\mathrm{D}}^2\left[1+v_{1\mathrm{D}}^2\right]$$

Refraction revisited



- Identify θ with $v_{\rm 1D}$

$$v_{\rm 1D} \leftrightarrow \frac{1}{\tan\theta} \quad \text{and} \quad v_{\rm 1D}^{\rm eff} \leftrightarrow \frac{1}{\tan\theta^{\rm eff}}$$

- Recall "Snell's law"

$$\frac{v_{\rm 2D}^{\rm eff}}{v_{\rm 2D}} = \frac{\sin\theta^{\rm eff}}{\sin\theta} \quad \Rightarrow \quad (v_{\rm 2D}^{\rm eff})^2\left[1+(v_{\rm 1D}^{\rm eff})^2\right] = v_{\rm 2D}^2\left[1+v_{\rm 1D}^2\right]$$

- Identify line density $\tilde{\rho}(v,\theta;0)$ with DOS $\rho_{\rm 1D}(p)$

$$\int \tilde{\rho}(v,\theta;0){\rm d}v{\rm d}\theta = \int \rho_{\rm 1D}(p){\rm d}p$$

$$\Rightarrow \quad \rho_{\rm 1D}(p) = \frac{v_{\rm 1D}'(p)}{1+v_{\rm 1D}^2(p)} \int \tilde{\rho}(v,{\rm acot}\ p;0){\rm d}v$$

Perspectives

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- Correlations, entropy, σ_{2D} ?

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- Correlations, entropy, $\sigma_{2\mathrm{D}}$?

- For any (1+1)D integrable model there should be an associated gas of lines. How much is this relevant ?