Introduction to Generalised Hydrodynamics in integrable field theories

Disordered Systems Advanced Lectures Series
4th lecture

Thibault Bonnemain, 26th February 2024

Recap: KdV

- KdV: integrable, nonlinear, dispersive PDE

$$\partial_t u + 6u\partial_x u + \partial_x^3 u = 0 \ .$$

- Infinite set of conservation laws

Time conserved "charges"

$$Q_n = \int \mathrm{d}x \ q_n(x,t) \ , \quad \text{and} \quad J_n = \int \mathrm{d}t \ j_n(x,t) \ ,$$

Space conserved "currents"

$$\boxed{\partial_t q_n + \partial_x j_n = 0 \ .}$$

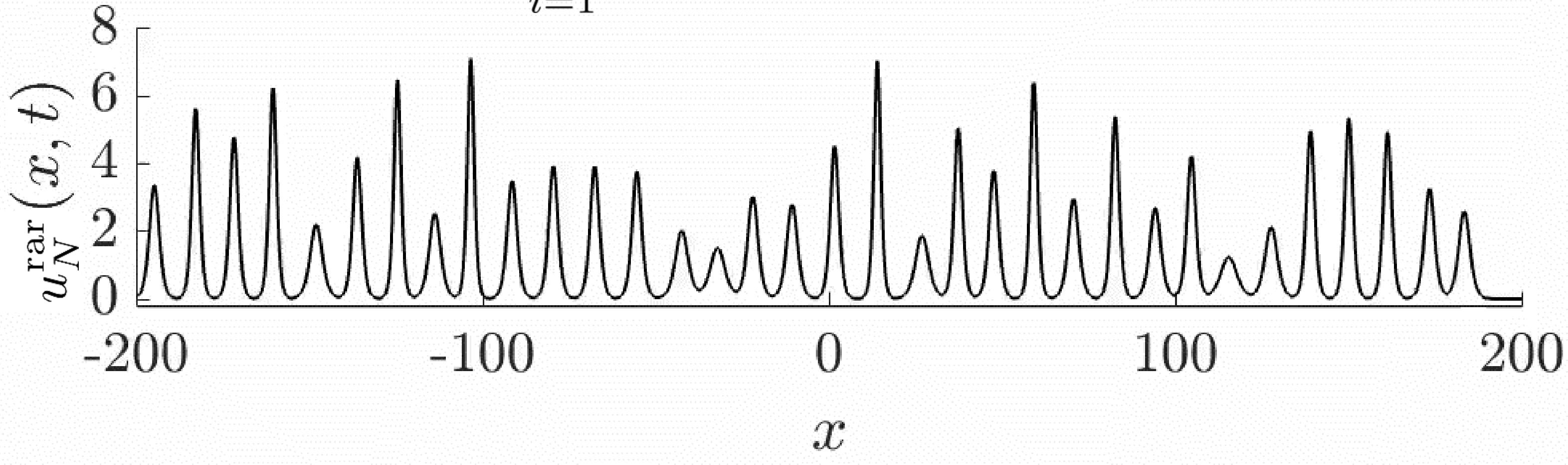
- Features $N-$soliton solutions

$$u_N(x,t) \approx \sum_{i=1}^{N} 2\eta_i^2 \mathrm{sech}^2 \left[\eta_i \left(x - 4\eta_i^2 t - x_i^{\pm}\right)\right] \quad \text{as} \quad t \to \pm\infty.$$

Recap: Zakharov's rarefied gas (1971)

- Random solution that almost everywhere in time can be approximated by

$$u_N^{\mathrm{rar}}(x,t) \approx \sum_{i=1}^{N} 2\eta_i^2 \mathrm{sech}^2\left[\eta_i\left(x - 4\eta_i^2 t - x_i\right)\right] \; .$$



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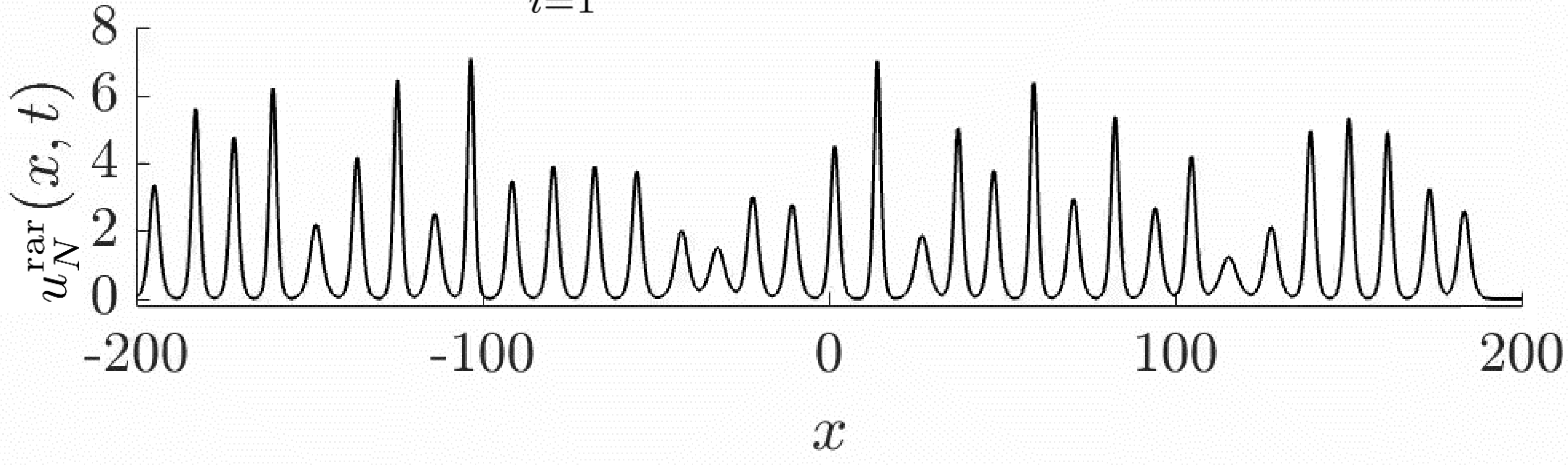

- Spectral support: $\Gamma = \bigcup_{i=0}^{N} [\gamma_{2i}, \gamma_{2i+1}] \subset \mathbb{R}^+$.

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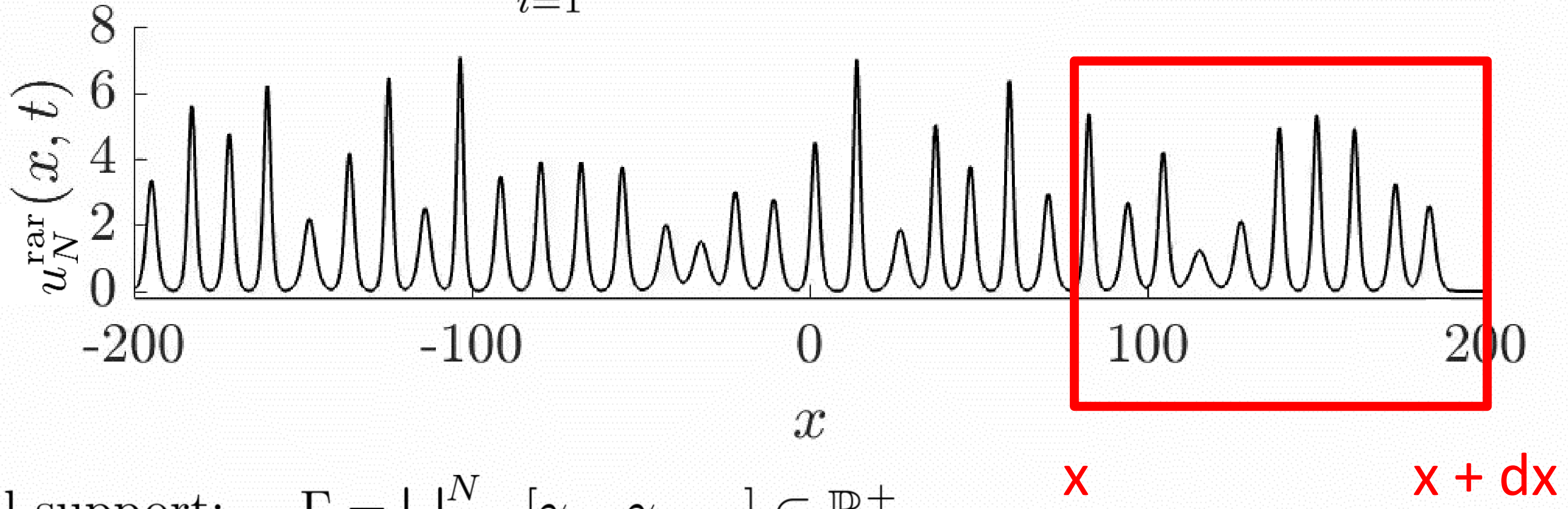

- Spectral support: $\Gamma = \bigcup_{i=0}^{N} [\gamma_{2i}, \gamma_{2i+1}] \subset \mathbb{R}^+$.
- Spectral density of states (DOS): $\rho^{\mathrm{rar}} : \Gamma \times \mathbb{R}^2 \to \mathbb{R}^+$

$$\rho^{\mathrm{rar}}(\eta; x, t)\mathrm{d}\eta\mathrm{d}x \ = \ \#\ \text{of solitons at } t \text{ in } [\eta, \eta + \mathrm{d}\eta] \times [x, x + \mathrm{d}x]$$

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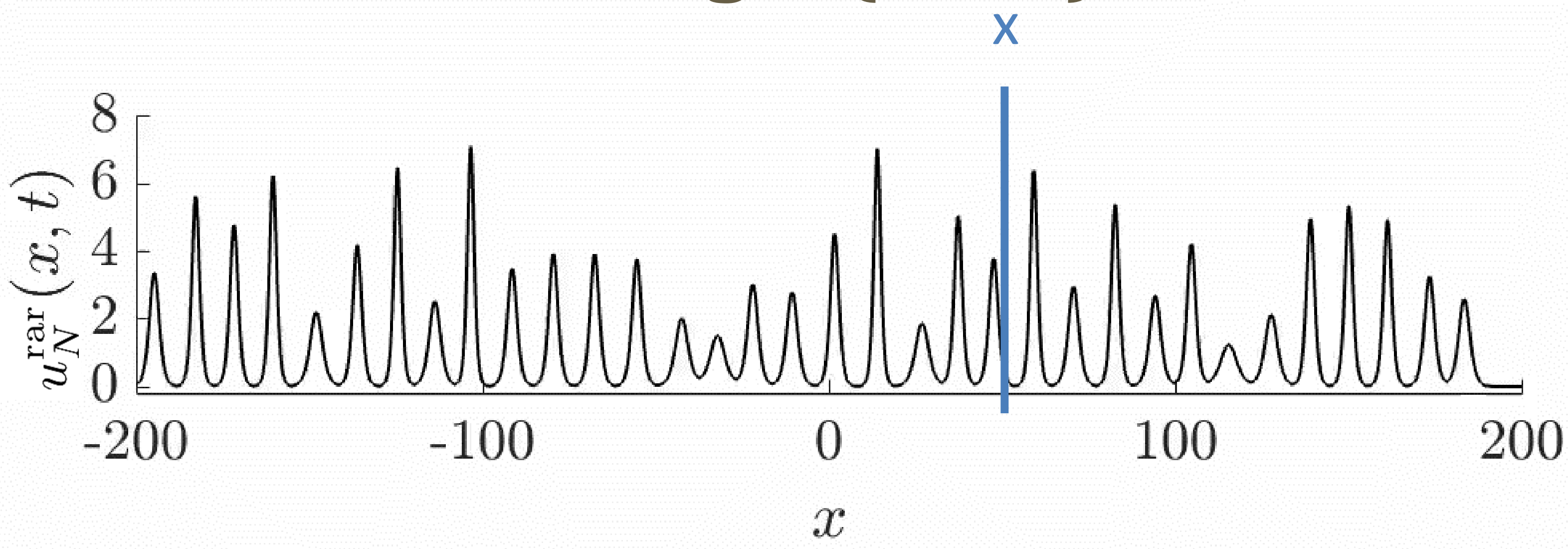

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- Spectral flux density: $f^{\rm rar} : \Gamma \times \mathbb{R}^2 \to \mathbb{R}$

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Recap: Zakharov's rarefied gas (1971)

- Isospectrality imposes DOS is only transported over large scales

$$\partial_t \rho^{\mathrm{rar}}(\eta; x, t) + \partial_x f(\eta; x, t) = 0 \ .$$

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- Isospectrality imposes DOS is only transported over large scales

$$\partial_t \rho^{\mathrm{rar}}(\eta; x, t) + \partial_x \left[v^{\mathrm{rar}}(\eta; x, t) \rho^{\mathrm{rar}}(\eta; x, t) \right] = 0 \ .$$

- Solitons move with effective velocity

$$\boxed{v^{\mathrm{rar}}(\eta; x, t) \approx 4\eta^2 + \frac{1}{\eta} \int_\Gamma \log \left| \frac{\eta + \mu}{\eta - \mu} \right| \rho^{\mathrm{rar}}(\eta; x, t) \left[4\eta^2 - 4\mu^2 \right] \mathrm{d}\mu \ .}$$

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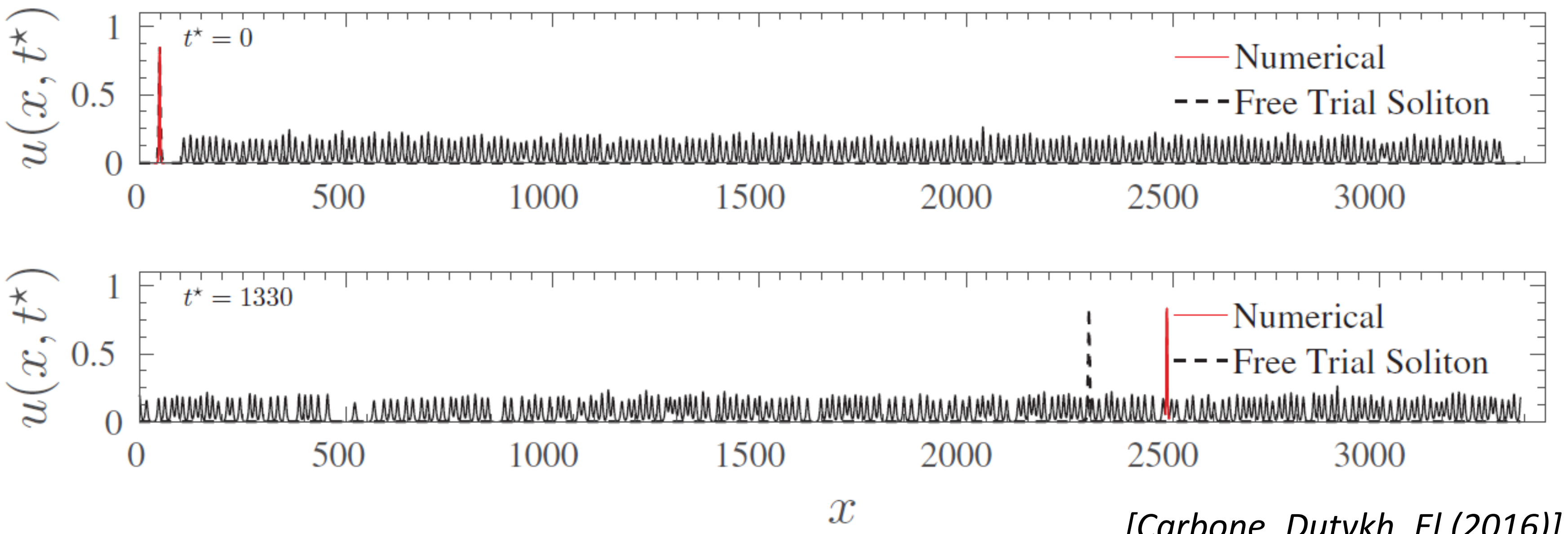
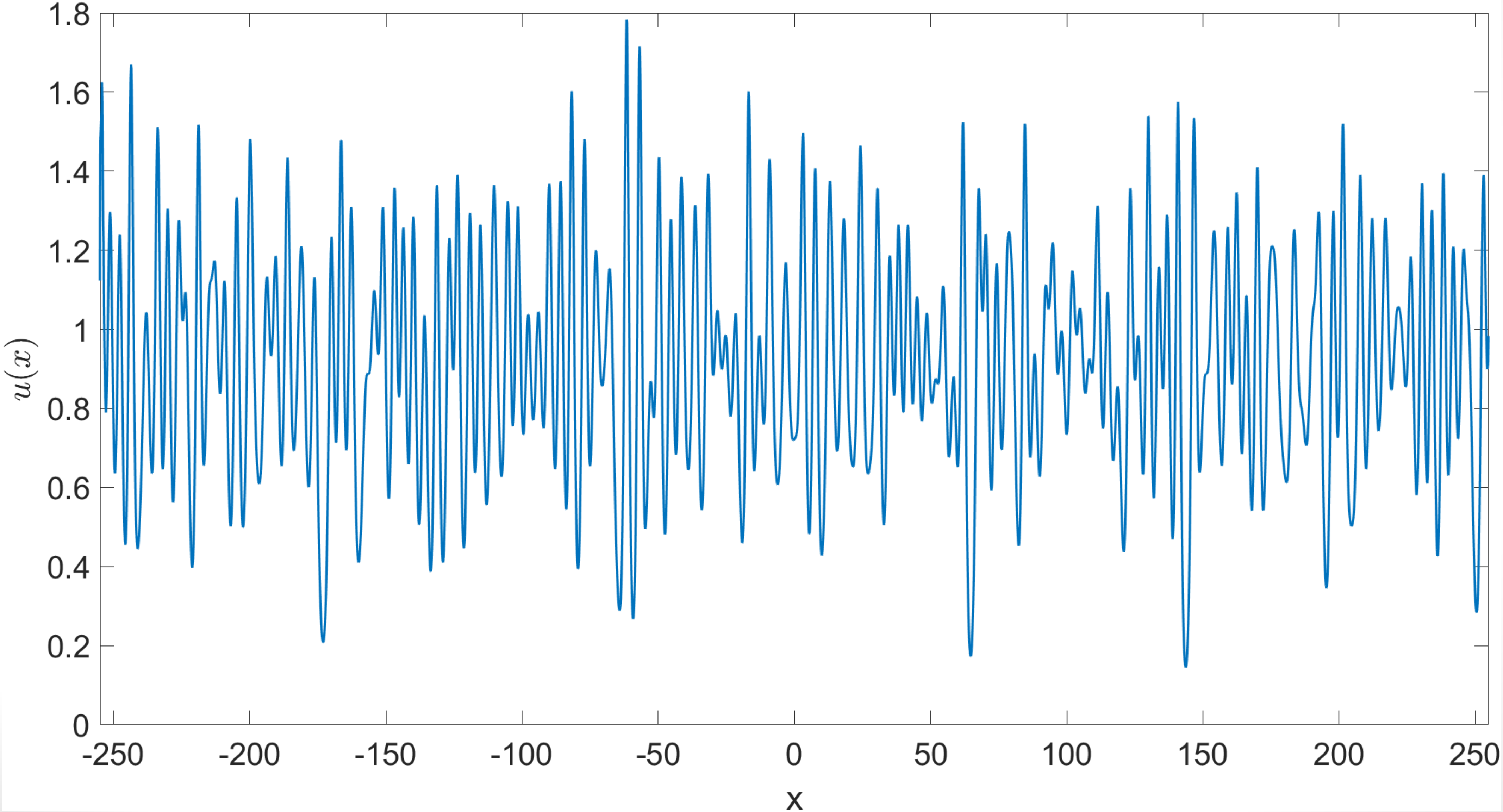


[Carbone, Dutykh, El (2016)]

Recap: dense gas



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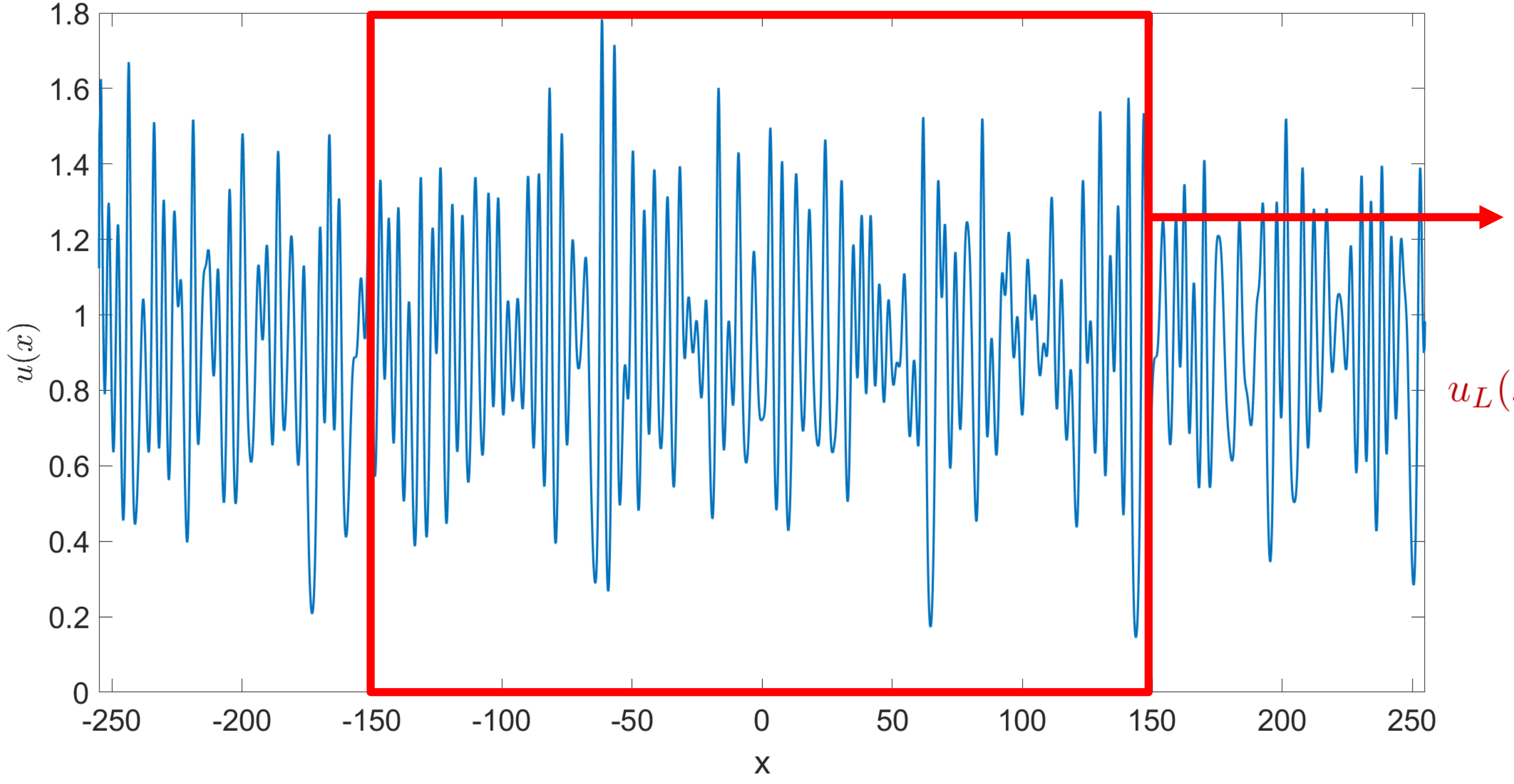

Fluid cell of size L containing N solitons

$$u_L(x,0) = \begin{cases} u(x)\ , & |x| < \dfrac{L}{2} \\ 0\ , & \text{otherwise} \end{cases}$$

$$\text{Asymptotically:} \quad u_L(x,t) \approx \sum_{i=1}^{N} 2\eta_i^2 \operatorname{sech}^2\left[\eta_i\left(x - 4\eta_i^2 t - x_i^{\pm}\right)\right] \quad \text{as} \quad t \to \pm\infty.$$

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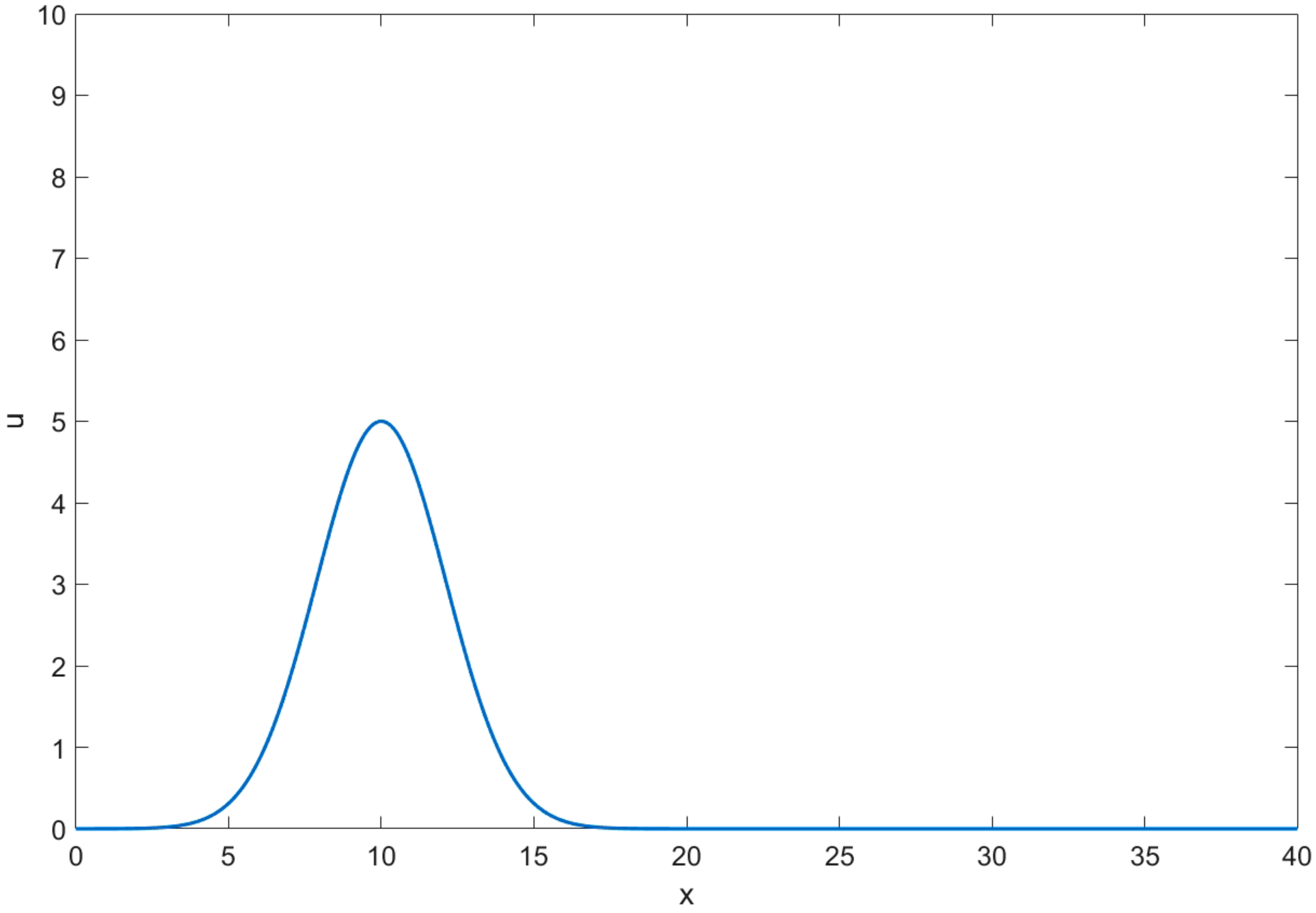

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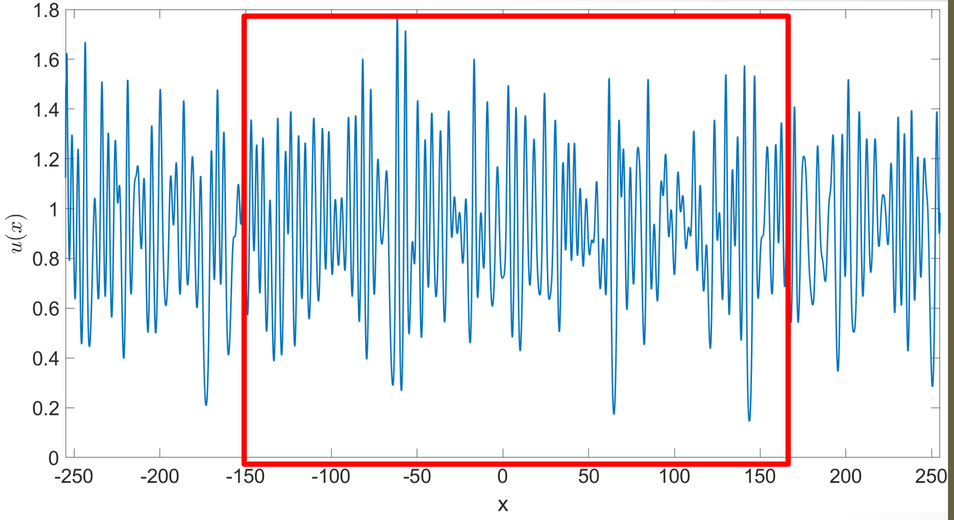

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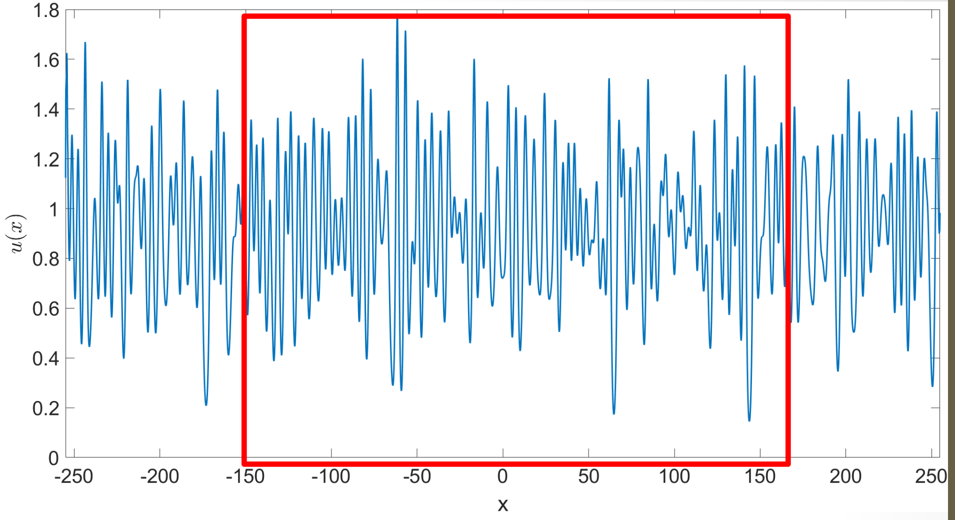

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- Change of metric and asymptotic space density

$$\frac{\mathrm{d}x^-(\eta)}{\mathrm{d}x} \equiv \mathcal{K}(\eta) = 1 - \frac{1}{\eta}\int_\Gamma \mathrm{d}\mu\, \rho(\mu) \log\left|\frac{\eta+\mu}{\eta-\mu}\right| .$$

Recap: thermodynamics

- Thermodynamic free energy density

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- Thermodynic averages

$$\langle q_n \rangle = \int_\Gamma \mathrm{d}\eta \; \rho(\eta) h_n(\eta) \; ,$$

- Static covariance matrix

$$\begin{aligned} \mathsf{C}_{ab} &\equiv \int_{\mathbb{R}} \mathrm{d}x \left(\langle q_a(x) q_b(0) \rangle - \langle q_a(x) \rangle \langle q_b(0) \rangle \right) \; , \\ &= \int_\Gamma \mathrm{d}\eta \, \rho(\eta) h_a^{\mathrm{dr}}(\eta) h_b^{\mathrm{dr}}(\eta) \; . \end{aligned}$$

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- Let $f : \Gamma \to \mathbb{R}$, we define the dressed function $f^{\mathrm{dr}} : \Gamma \to \mathbb{R}$ from this Fredholm equation of the 2nd kind

$$f^{\mathrm{dr}}(\eta) = f(\eta) + \int_\Gamma \frac{\mathrm{d}\mu}{2\pi} \, 8 \log \left| \frac{\eta - \mu}{\eta + \mu} \right| n(\mu) f^{\mathrm{dr}}(\mu) \; .$$

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$$\eta \mathcal{K}(\eta) = \eta^{\mathrm{dr}}(\eta)$$

Recap: nonlinear dispersion relations

- N-phase solutions associated with band spectrum $\lambda \in [\lambda_0, \lambda_1] \cup \cdots \cup [\lambda_{2N}, +\infty[$

Riemann Theta

Riemann period matrix

$$u(x,t) = \Lambda + \Phi - 2\log\left[\Theta\left(\boldsymbol{\theta}(x,t);\mathbf{B}\right)\right]_{xx} \quad .$$

Constants (depend on endpoints)

Phase vector

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- Nonlinear dispersion relations (NDRs)

$$\boxed{\mathbf{k} = 4\pi i \mathbf{B}^{-1}\mathbf{c}^{(N)} \; , \quad \text{and} \quad \boldsymbol{\omega} = 8\pi i \mathbf{B}^{-1}\left[\Lambda \mathbf{c}^{(N)} + 2\mathbf{c}^{(N-1)}\right] \; ,}$$

with $[\mathbf{c}^{(M)}]_j = c_{jM}$.

Recap: nonlinear dispersion relations

- Thermodynamic spectral limit

  - solitonic limit: $\lambda_{2j} \to -\eta_j^2 \ , \quad \text{and} \quad \lambda_{2j+1} \to -\eta_j^2 \ , \quad j = 1, 2 \cdots, N.$

  - $N \to \infty$: $k_j \to 0 \ , \quad \omega_j \to 0 \ , \quad \text{while} \quad \frac{1}{2\pi}\sum_{j=1}^{N} k_j = \alpha \ , \quad \frac{1}{2\pi}\sum_{j=1}^{N} \omega_j = \beta \ .$

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- Thermodynamic NDRs with the spectral scaling function $\sigma(\eta) > 0$

$$\frac{1}{2\pi}\sum_{j=1}^{M<N} k_j \to \int_{\eta_0}^{\eta} \rho(\mu)\mathrm{d}\mu , \qquad \int_{\Gamma} \log\left|\frac{\eta+\mu}{\eta-\mu}\right| \rho(\mu)\mathrm{d}\mu + \sigma(\eta)\rho(\eta) = \eta ,$$

$$\frac{1}{2\pi}\sum_{j=1}^{M<N} \omega_j \to \int_{\eta_0}^{\eta} f(\mu)\mathrm{d}\mu , \qquad \int_{\Gamma} \log\left|\frac{\eta+\mu}{\eta-\mu}\right| f(\mu)\mathrm{d}\mu + \sigma(\eta)f(\eta) = 4\eta^3 .$$

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$$
\begin{aligned}
\int_\Gamma \log\left|\frac{\eta+\mu}{\eta-\mu}\right| \rho(\mu)\mathrm{d}\mu \ + \ \sigma(\eta)\rho(\eta) = \eta \quad &\Rightarrow \quad \eta\mathcal{K}(\eta) = (\eta)^{\mathrm{dr}}(\eta) = \sigma(\eta)\rho(\eta) \ , \\
\int_\Gamma \log\left|\frac{\eta+\mu}{\eta-\mu}\right| f(\mu)\mathrm{d}\mu \ + \ \sigma(\eta)f(\eta) = 4\eta^3 \quad &\Rightarrow \quad (4\eta^3)^{\mathrm{dr}}(\eta) = \sigma(\eta)f(\eta) \ .
\end{aligned}
$$

$$\Rightarrow \quad \sigma(\eta) = \frac{\pi}{4n(\eta)}$$

Outline of the lectures

I. **Elements of Hydrodynamics**

II. **Integrable field theories**

III. **Soliton gas and Generalised Hydrodynamics**

   4) (Generalised) Hydrodynamics of the KdV gas.

IV. **Specific examples and potential extensions**

   1) Illustration: polychromatic solitons gases.
   2) Illustration: soliton condensates.
   3) ~~Generating N-soliton solutions numerically.~~
   4) Extensions of GHD in KdV.
   5) Application of GHD to other models.

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Note that in rarefied gas $\int_\Gamma \rho(\mu)\mathrm{d}\mu = \alpha \ll 1$

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- Generic form of the effective velocity in GHD

$$v^{\text{eff}}(\eta) = v^{\text{gr}}(\eta) + \int_\Gamma \mathrm{d}\mu\ \varphi(\eta;\mu)\rho(\mu)\left[v^{\text{eff}}(\eta) - v^{\text{eff}}(\mu)\right] \ ,$$

$$v^{\text{gr}}(\eta) = \frac{E'(\eta)}{p'(\eta)} \ , \qquad v^{\text{eff}}(\eta) = \frac{(E')^{\text{dr}}(\eta)}{(p')^{\text{dr}}(\eta)} \ .$$

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$$v^{\text{gr}}(\eta) = \frac{E'(\eta)}{p'(\eta)} \overset{\text{KdV}}{=} \frac{32\eta^3}{8\eta} \ , \qquad v^{\text{eff}}(\eta) = \frac{(E')^{\text{dr}}(\eta)}{(p')^{\text{dr}}(\eta)} \overset{\text{KdV}}{=} \frac{(4\eta^3)^{\text{dr}}(\eta)}{(\eta)^{\text{dr}}(\eta)} = \frac{f(\eta)}{\rho(\eta)} \ .$$

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$$\Rightarrow \quad \boxed{\partial_t \rho(\eta;x,t) + \partial_x \left[v^{\mathrm{eff}}(\eta;x,t) \rho(\eta;x,t) \right] = 0 \ .}$$

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From thermodynamics to hydrodynamics

- Conservation of waves

$$\partial_t \mathbf{k} + \partial_x \boldsymbol{\omega} = 0 \ .$$

- Slow modulations of finite gap solutions

$$\mathbf{k} = \mathbf{K}[\boldsymbol{\lambda}(x,t)] \ , \quad \boldsymbol{\omega} = \boldsymbol{\Omega}[\boldsymbol{\lambda}(x,t)] \ .$$

- Thermodynamic spectral limit and leading order in multi-scale expansion

$$\boxed{\begin{gathered}\partial_t \rho(\eta;x,t) + \partial_x \left[v^{\mathrm{eff}}(\eta;x,t)\rho(\eta;x,t)\right] = 0 \ , \\ v^{\mathrm{eff}}(\eta;x,t) = 4\eta^2 + \frac{1}{\eta}\int_\Gamma \log\left|\frac{\eta+\mu}{\eta-\mu}\right| \rho(\eta;x,t)[v^{\mathrm{eff}}(\eta;x,t) - v^{\mathrm{eff}}(\eta;x,t)]\mathrm{d}\mu \ .\end{gathered}}$$

Alternative derivation of GHD equations

[Based on: Doyon, Spohn, Yoshimura (2017)]

- Asymptotic dynamics

$$x_j^-(t) = x_j^-(0) + 4\eta_j^2 t \ ,$$

$$\Rightarrow \quad \partial_t \rho^-(\eta; x^-, t) + 4\eta^2 \partial_{x^-} \rho^-(\eta; x^-, t) = 0 \ .$$

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- Change of metric: $\mathrm{d}x^-(\eta; x, t) = \mathcal{K}(\eta; x, t)\mathrm{d}x$

$$\partial_t n(\eta; x, t) + v^{\mathrm{eff}}(\eta; x, t)\partial_x n(\eta; x, t) = 0 \ .$$

$$v^{\mathrm{eff}}(\eta; x, t) = 4\eta^2 + \frac{1}{\eta}\int_\Gamma \log\left|\frac{\eta+\mu}{\eta-\mu}\right| \rho(\eta; x, t)[v^{\mathrm{eff}}(\eta; x, t) - v^{\mathrm{eff}}(\eta; x, t)]\mathrm{d}\mu \ .$$

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n (equivalently σ) plays the role of a continuum of Riemann invariants!

GHD as an integrable system of hydrodynamic type

- System of hydrodynamic type in Riemann form

$$\partial_t \lambda_j + v_j(\lambda_0, \cdots, \lambda_n) \partial_x \lambda_j = 0 \quad \longrightarrow \quad \partial_t n(\eta; x, t) + v^{\mathrm{eff}}(\eta; x, t) \partial_x \left[n(\eta; x, t) \right] = 0 \, .$$

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- Linear degeneracy

$$\partial_{\lambda_j} v_j = 0 \quad \longrightarrow \quad \frac{\delta v^{\mathrm{eff}}(\eta)}{\delta n(\eta)} = 0\,, \quad \forall \eta \in \Gamma\,.$$

No shocks in GHD!

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No shocks in GHD!

- Semi-Hamiltonian property

$$\partial_{\lambda_j} \frac{\partial_{\lambda_k} v_i}{v_k - v_i} = \partial_{\lambda_k} \frac{\partial_{\lambda_j} v_i}{v_j - v_i} \,, \quad i \neq j \neq k$$

$$\int_\Gamma \mathrm{d}\nu \left[\frac{\delta}{\delta n(\nu)} \left(\frac{\delta v^{\text{eff}}(\eta)/\delta n(\mu)}{v^{\text{eff}}(\mu) - v^{\text{eff}}(\eta)}\right)\right] = \int_\Gamma \mathrm{d}\mu \left[\frac{\delta}{\delta n(\mu)} \left(\frac{\delta v^{\text{eff}}(\eta)/\delta n(\nu)}{v^{\text{eff}}(\nu) - v^{\text{eff}}(\eta)}\right)\right] \,.$$

GHD equations are integrable!

GHD as an integrable system of hydrodynamic type

- System in Riemann form

$$\partial_t n(\eta;x,t)+v^{\mathrm{eff}}(\eta;x,t)\partial_x\left[n(\eta;x,t)\right]=0\,.$$

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GHD equations are integrable!

- Generalised hodograph transform

$$x-4\eta^2t=\int_{n(\eta;0,0)}^{n(\eta;x,t)}\zeta g(\zeta;\eta)\mathrm{d}\zeta+\int_\Gamma \mathrm{d}\mu\frac{1}{\mu}\log\left|\frac{\eta+\mu}{\eta-\mu}\right|\int_{n(\mu;0,0)}^{n(\mu;x,t)}g(\zeta;\mu)\mathrm{d}\zeta\,,$$

where $g(\zeta;\eta)$ are functional degrees of freedom.

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Soliton Gas theory

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GHD

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Illustration: polychromatic soliton gas

[El, Kamchatnov, Pavlov, Zykov (2011)]

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- DOS as a sum of weighted Dirac delta-functions

$$\rho(\eta; x, t) = \sum_{i=1}^{M} w_j(x,t)\delta(\eta_j - \eta) \ .$$

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$$\partial_t w_j + \partial_x \left[V_j w_j\right] = 0 \ ,$$

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$$\partial_t r_j + V_j \partial_x r_j = 0 \ ,$$

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$$\left.\begin{aligned} \partial_t r_j + V_j \partial_x r_j = 0 \\ r_j = \frac{1}{\eta_j w_j}\left[1 - \frac{1}{\eta_j}\sum_{k\neq j} \log\left|\frac{\eta_j + \eta_k}{\eta_j - \eta_k}\right| w_k\right] \end{aligned}\right\} \longleftrightarrow \left\{\begin{aligned} &\partial_t \sigma + v^{\mathrm{eff}} \partial_x \sigma = 0 \\ &\sigma(\eta)\rho(\eta) = 1 - \frac{1}{\eta}\int_\Gamma \log\left|\frac{\eta+\mu}{\eta-\mu}\right| \rho(\mu) \mathrm{d}\mu \end{aligned}\right.$$

$$r_j \equiv \sigma(\eta_j)/\eta_j^2$$

Illustration: polychromatic soliton gas

[El, Kamchatnov, Pavlov, Zykov (2011)]

- For any finite M, the system is linearly degenerate and semi-Hamiltonian

$$\partial_{r_j} V_j = 0 \qquad \text{and} \qquad \partial_{r_j} \frac{\partial_{r_k} V_i}{V_k - V_i} = \partial_{r_k} \frac{\partial_{r_j} V_i}{V_j - V_i} \ ,$$

hence integrable via hodograph transform

$$x - 4\eta_j^2 t = \int^{r_j} \mathrm{d}\xi \ \xi g_j(\xi) + \frac{1}{\eta_j} \sum_{k \neq j} \log \left| \frac{\eta_j + \eta_k}{\eta_j - \eta_k} \right| \int^{r_k} \mathrm{d}\xi \ g_k(\xi) \ .$$

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- For any finite M the polychromatic reduction is a Liouville integrable Hamiltonian system

[Based on: Bulchandani (2017)]

$$\partial_t r_j = \{r_j ; \mathcal{H}\} \qquad \text{with} \qquad \mathcal{H} = -\sum_{j=1}^{M} \int_{\mathbb{R}} \mathrm{d}x \ r_j^4 \ w_j^2 \ V_j \ .$$

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$\Rightarrow$ GHD equations should be an infinite dimensional integrable Hamiltonian system.

Illustration: 2 component Riemann problem

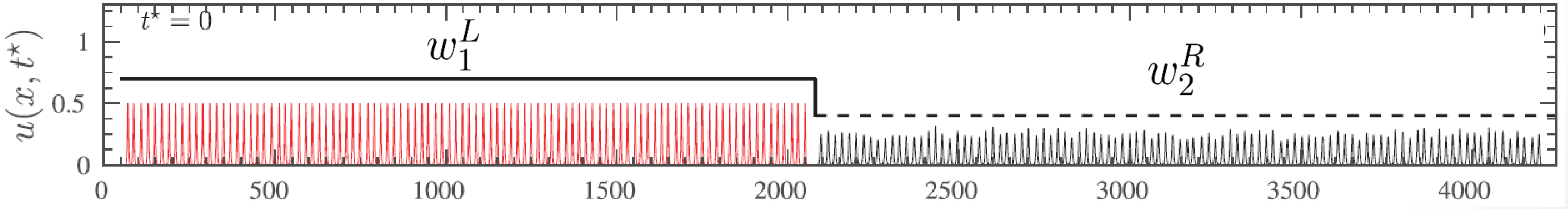
- Riemann problem (a.k.a partitioning protocol, shock tube problem ...), $\eta_1 > \eta_2$

$$\partial_t w_j + \partial_x \left[V_j w_j \right] = 0 \ ,$$

$$V_j = 4\eta_j^2 + \frac{1}{\eta_j} \log \left| \frac{\eta_1 + \eta_2}{\eta_1 - \eta_2} \right| w_{3-j} (V_j - V_{3-j}) \ .$$

with

$$w_1(x, t=0) = \begin{cases} w_1^L \ , & \text{if } x < 0 \\ 0 \ , & \text{if } x \geq 0 \end{cases} \quad \text{and} \quad w_2(x, t=0) = \begin{cases} 0 \ , & \text{if } x < 0 \\ w_2^R \ , & \text{if } x \geq 0 \end{cases} \ .$$



[Carbone, Dutykh, El (2016)]

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- Characteristic velocities

$$V_j = 4\eta_j^2 + \frac{4\varphi w_{3-j}(\eta_j^2 - \eta_{3-j}^2)}{1 - \varphi(w_{3-j}/\eta_j - w_j/\eta_{3-j})} \ .$$

Illustration: 2 component Riemann problem

- Linear degeneracy and scale invariance under $x \to Cx$ and $t \to Ct$

$$w_j(x,t) = \begin{cases} w_j^L \ , & \text{if} \quad x/t < c^L \ , \\ w_j^C \ , & \text{if} \quad c^L \leq x/t < c^R \ , \\ w_j^R \ , & \text{if} \quad c^R \leq x/t \ . \end{cases}$$

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- Conservation of the number of solitons accross the discontinuities (Rankine-Hugoniot)

$$\lim_{\epsilon \to 0} \int_{c^{\cdot} t - \epsilon}^{c^{\cdot} t + \epsilon} \mathrm{d}x \left[\partial_t w_j + \partial_x \left(V_j w_j \right) \right] = 0 \quad \Rightarrow \quad \begin{cases} c^L (w_j^L - w_j^C) = w_j^L V_j^L - w_j^C V_j^C \ , \\ c^R (w_j^C - w_j^R) = w_j^C V_j^C - w_j^R V_j^R \ . \end{cases}$$

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- Conservation of the number of solitons accross the discontinuities (Rankine-Hugoniot)

$$w_1^C = \frac{w_1^L(1 - \varphi w_2^R/\eta_2)}{1 - \varphi^2 w_1^L w_2^R/\eta_1\eta_2}\ , \qquad w_2^C = \frac{w_2^R(1 - \varphi w_1^L/\eta_1)}{1 - \varphi^2 w_1^L w_2^R/\eta_1\eta_2}\ ,$$

$$c^L = 4\eta_2^2 - \frac{4(\eta_1^2 - \eta_2^2)\varphi w_1^C}{\eta_1 - \varphi w_1^C - \varphi w_2^C \eta_1/\eta_2}\ , \qquad c^R = 4\eta_1^2 + \frac{4(\eta_1^2 - \eta_2^2)\varphi w_2^C}{\eta_2 - \varphi w_2^C - \varphi w_1^C \eta_2/\eta_1}\ .$$

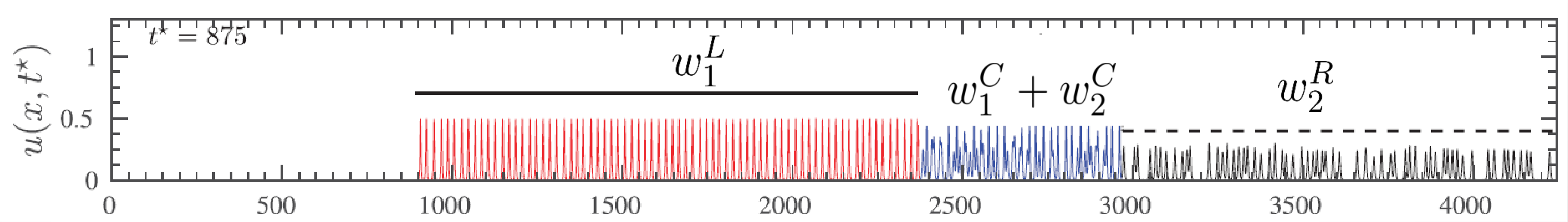

Illustration: soliton condensate

[Congy, El, Roberti, Tovbis (2023)]

- Soliton condensate: limit $\sigma \to 0$ of the NDRs

$$\int_\Gamma \log\left|\frac{\eta+\mu}{\eta-\mu}\right| \rho(\mu)\mathrm{d}\mu \; + \; \cancel{\sigma(\eta)\rho(\eta)} = \eta \; ,$$

$$\int_\Gamma \log\left|\frac{\eta+\mu}{\eta-\mu}\right| f(\mu)\mathrm{d}\mu \; + \; \cancel{\sigma(\eta)f(\eta)} = 4\eta^3 \; .$$

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$\Rightarrow$ densest gas.

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- DOS of a genus N condensate: $\Gamma = [0,\gamma_0] \cup [\gamma_1,\gamma_2] \cup \cdots [\gamma_{2N-1},\gamma_{2N}]$

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Odd continuation of ρ to $-\Gamma$.

$$-\int_{\tilde\Gamma \equiv -\Gamma\cup\Gamma} \log|\eta-\mu| \, \rho(\mu)\mathrm{d}\mu = \eta \ .$$

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∂_η

$$\int_{\tilde{\Gamma}} \frac{\rho(\mu)}{\mu-\eta}\mathrm{d}\mu = 1 \quad \Leftrightarrow \quad \underline{H[\rho]}(\eta) = \frac{1}{\pi} \; .$$

Finite Hilbert transform

Illustration: genus N DOS

[Congy, El, Roberti, Tovbis (2023)]

- General genus N solution

$$\rho_N(\eta) = \frac{iP(\eta)}{2\pi R(\eta)}$$

Monic polynomial of degree $2N+1$

$$R^2(\eta) = \prod_{j=1}^{2N+1} (\eta^2 - \gamma_j^2)$$

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$$P(0) = 0 \quad \text{and} \quad \int_{\gamma_{2j}}^{\gamma_{2j+1}} \rho_N(\eta) \mathrm{d}\eta = 0 \; .$$

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- Example: genus 0

$$\rho_0(\eta) = \frac{\eta}{\pi\sqrt{\gamma_0^2 - \eta^2}}$$

- Example: genus 1

$$\rho_1(\eta) = \frac{i\eta(\eta^2 - w^2)}{\pi R(\eta)} \; , \quad w = \gamma_2^2 - (\gamma_2^2 - \gamma_0^2)\frac{\mathrm{E}(m)}{\mathrm{K}(m)} \; ,$$

$$m = \frac{\gamma_1^2 - \gamma_0^2}{\gamma_1^2 - \gamma_2^2} \; .$$

Illustration: properties of the genus 0 condensate

[Congy, El, Roberti, Tovbis (2023)]

- Thermodynamic averages of the charge densities

$$\langle q_n \rangle = \int_0^{\gamma_0} \eta^{2n+1} \rho_0(\eta) \mathrm{d}\eta = \frac{\Gamma(3/2+n)}{2\sqrt{\pi}\,\Gamma(n+2)}\ \gamma_0^{2(n+1)} \ ,$$

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e.g.

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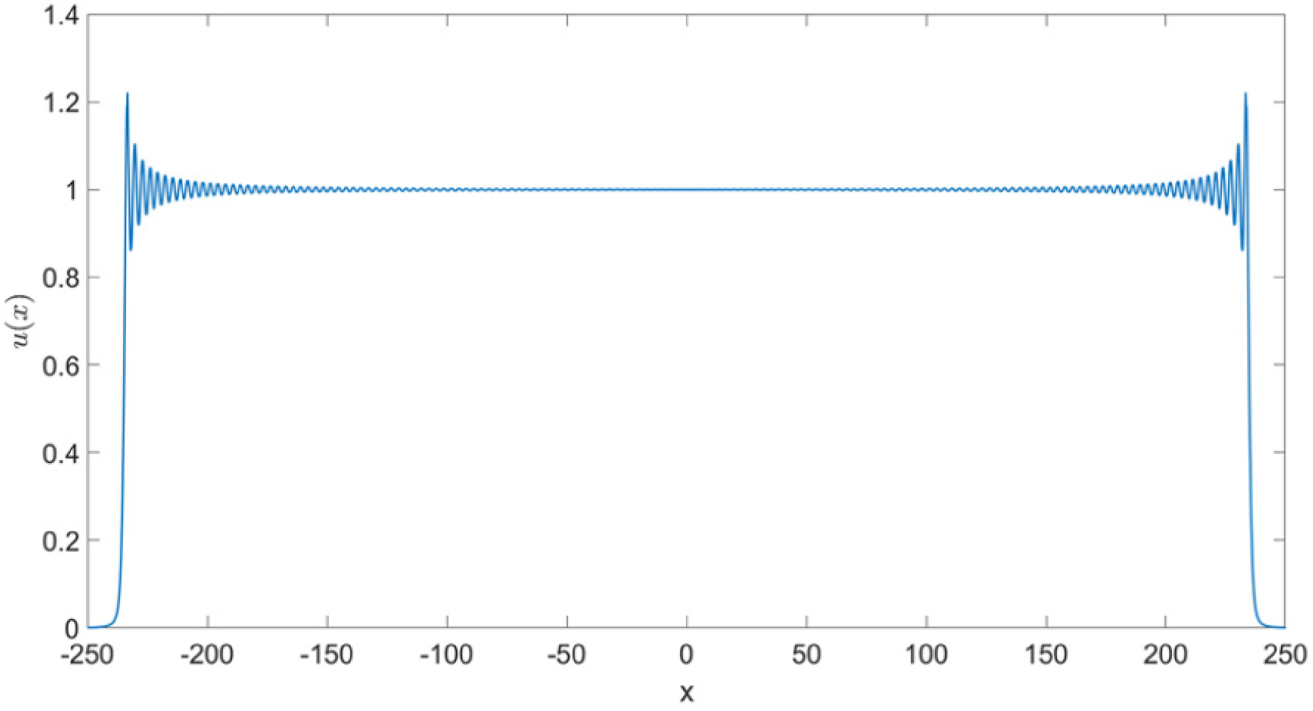

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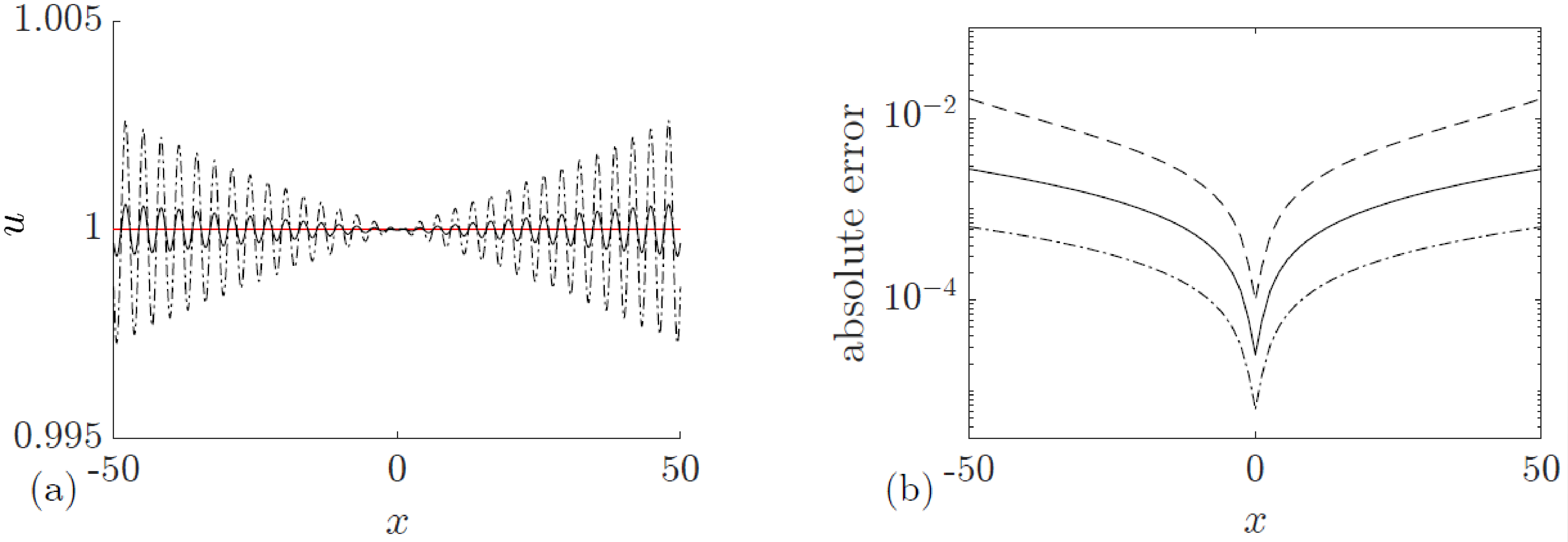
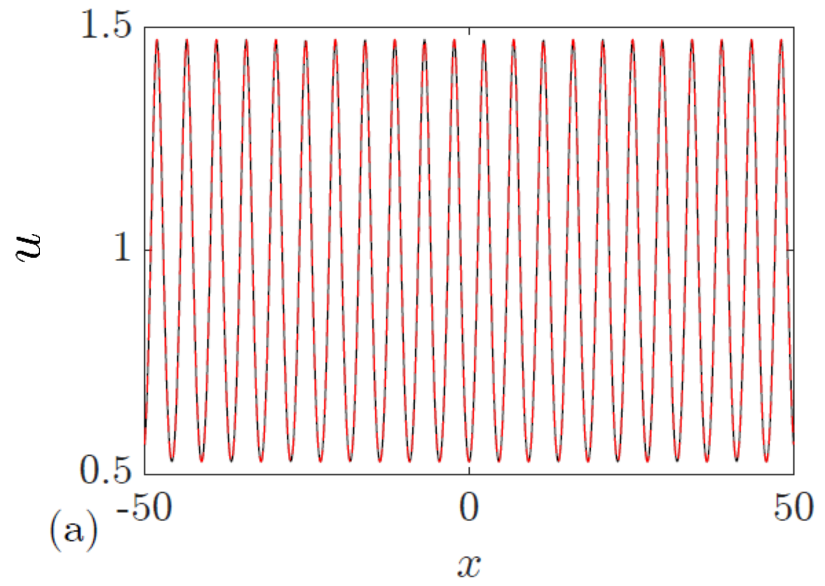
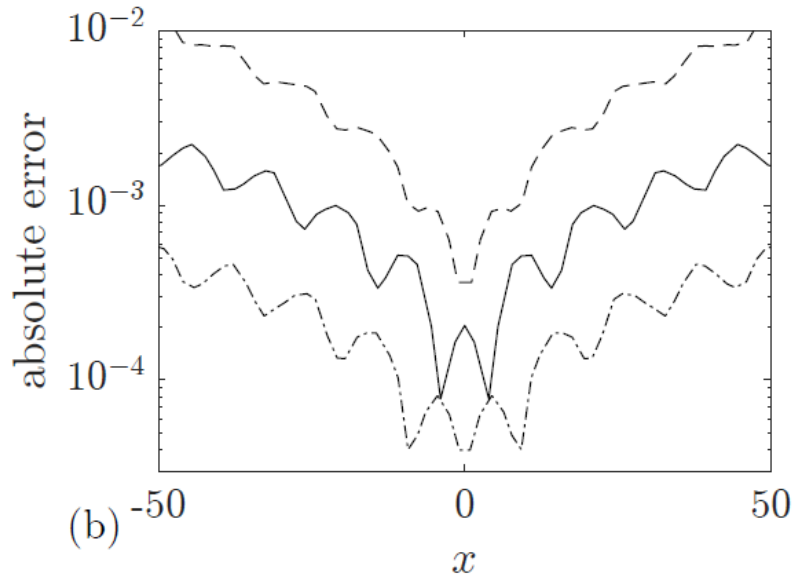
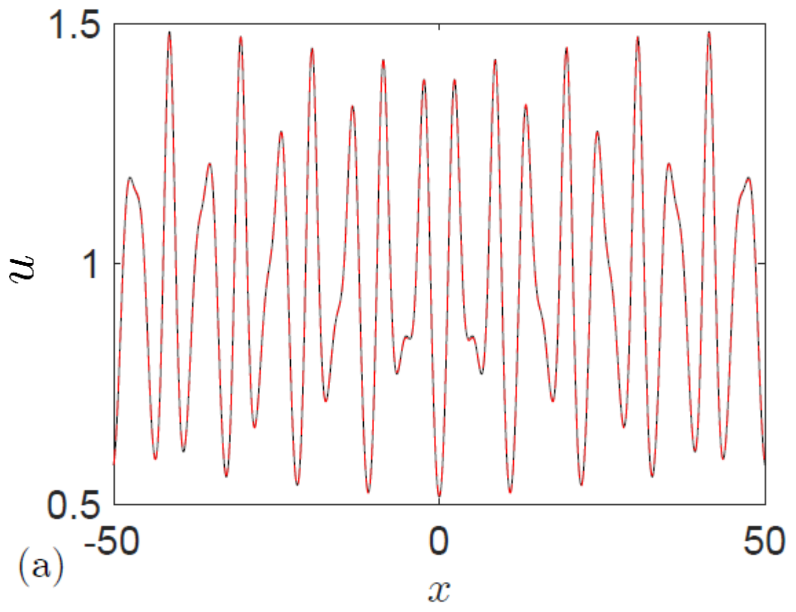


Illustration: genus 1 and 2 condensates

[Congy, El, Roberti, Tovbis (2023)]







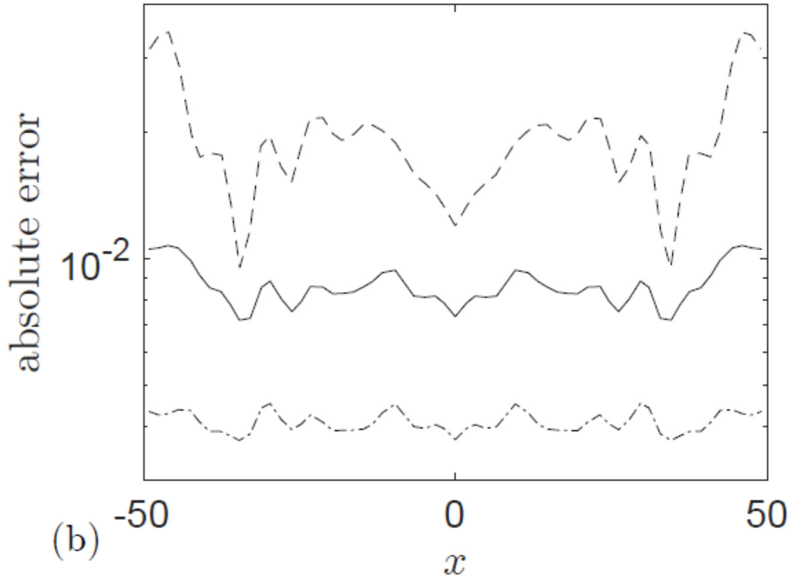


Illustration: hydrodynamics of the condensate

[Congy, El, Roberti, Tovbis (2023)]

- GHD equations for the KdV soliton gas

$$\partial_t \rho(\eta; x, t) + \partial_x \left[v^{\text{eff}}(\eta; x, t) \rho(\eta; x, t) \right] = 0 \ ,$$

$$v^{\text{eff}}(\eta; x, t) = 4\eta^2 + \frac{1}{\eta} \int_\Gamma \log \left| \frac{\eta + \mu}{\eta - \mu} \right| \rho(\eta; x, t) [v^{\text{eff}}(\eta; x, t) - v^{\text{eff}}(\eta; x, t)] \mathrm{d}\mu \ .$$

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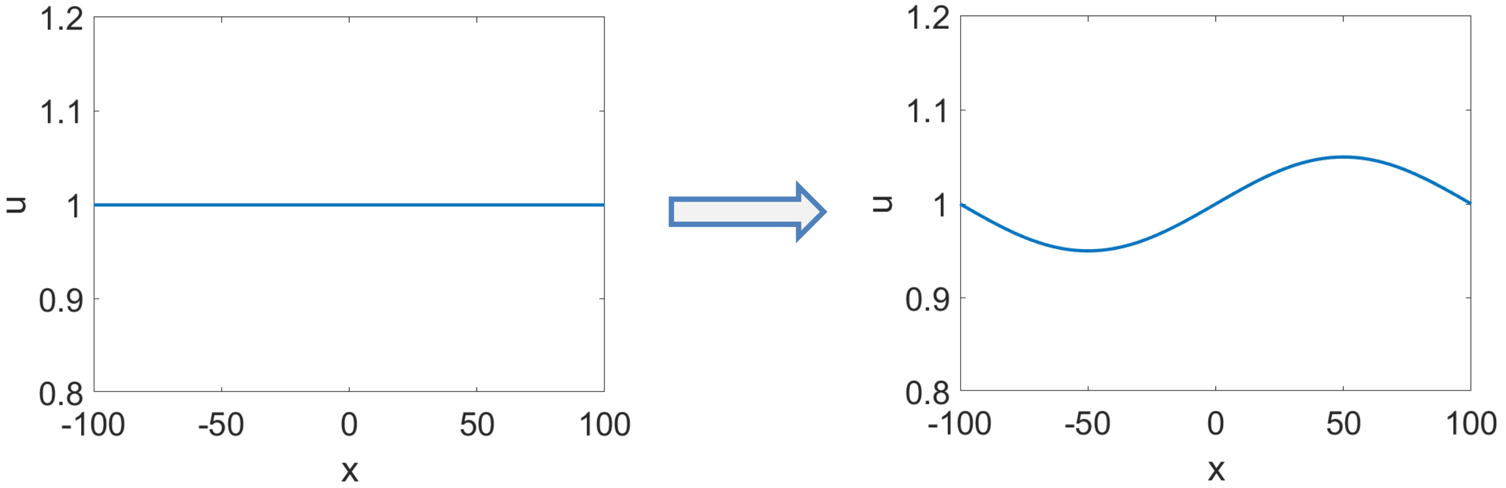

Condensate

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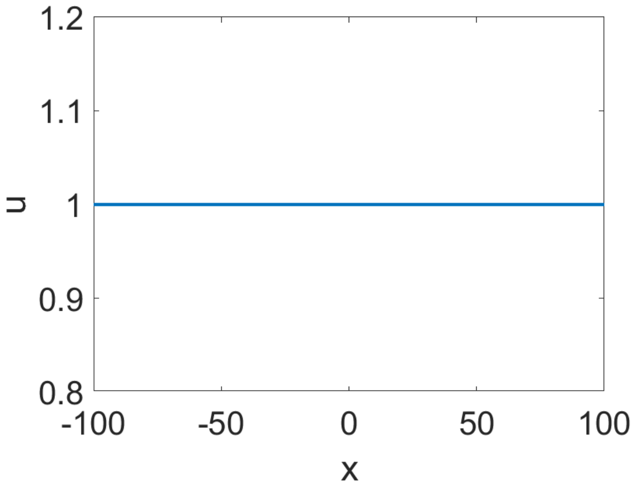

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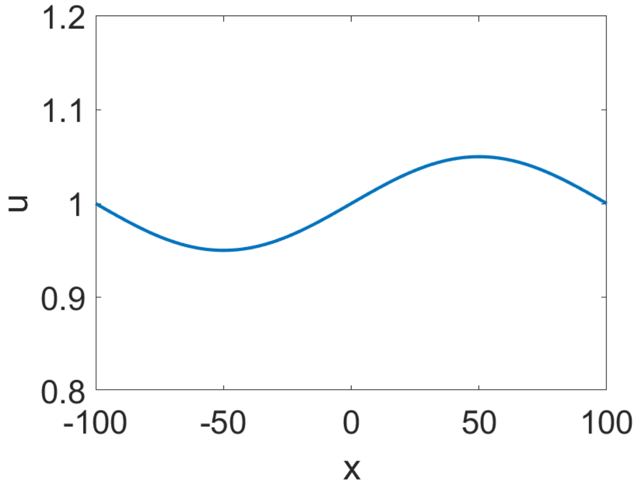

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- Shocks regularised by increasing the genus, e.g. Riemann problem

$$u(x, t=0) = \begin{cases} \gamma_-^2 \ , & x < 0 \\ \gamma_+^2 \ , & x > 0 \end{cases} \quad \Rightarrow \quad \rho(\eta; x, t=0) = \begin{cases} \dfrac{\eta}{\pi\sqrt{\gamma_-^2 - \eta^2}} \ , & x < 0 \\ \dfrac{\eta}{\pi\sqrt{\gamma_+^2 - \eta^2}} \ , & x > 0 \end{cases}$$

Illustration: condensate Riemann problem

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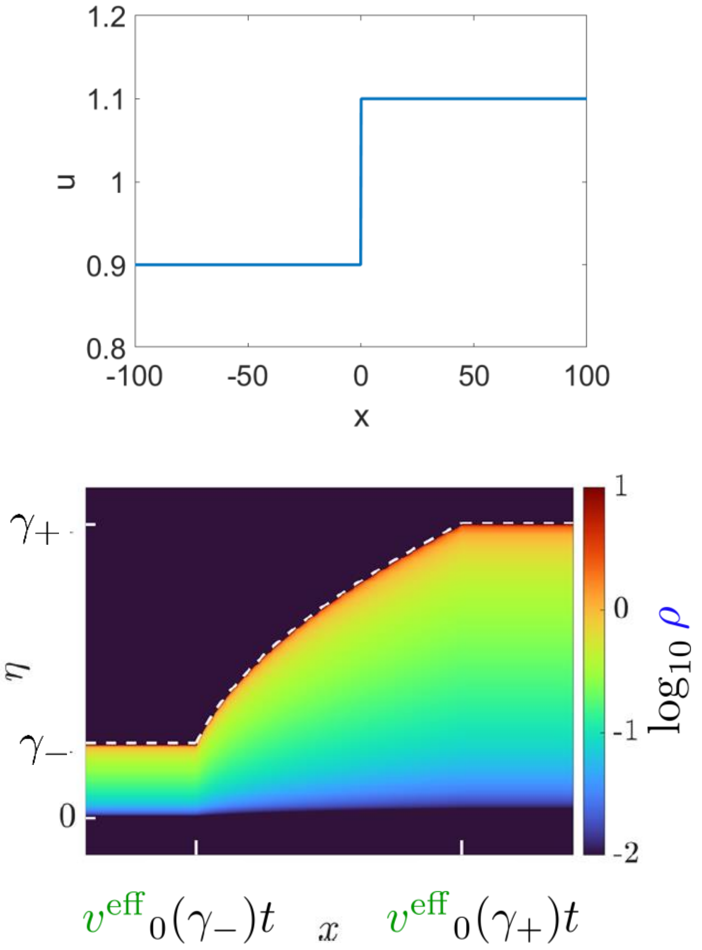

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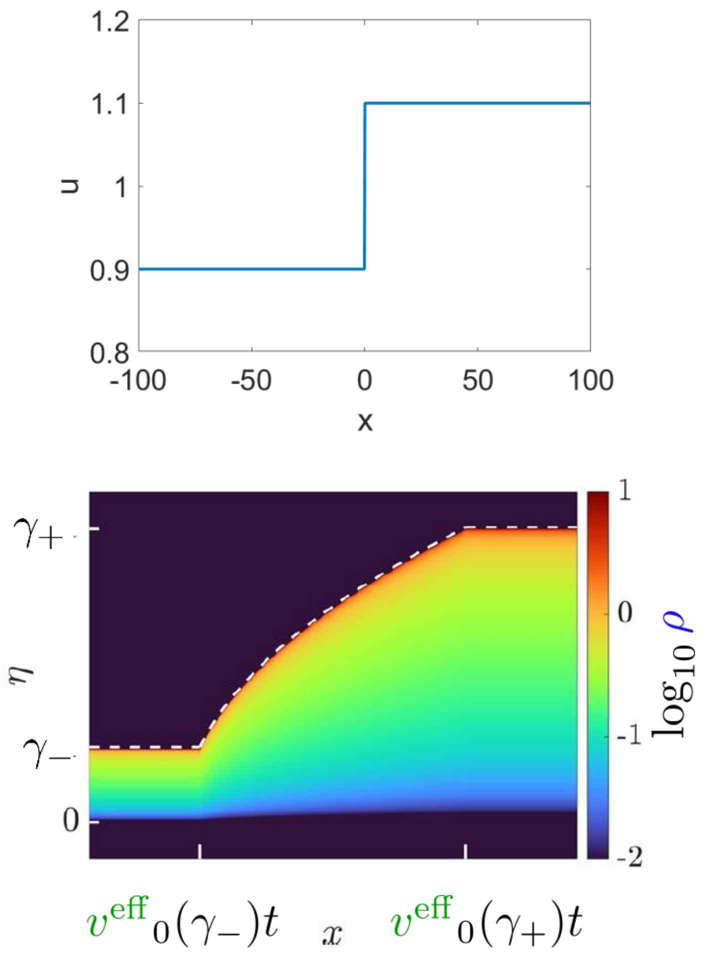

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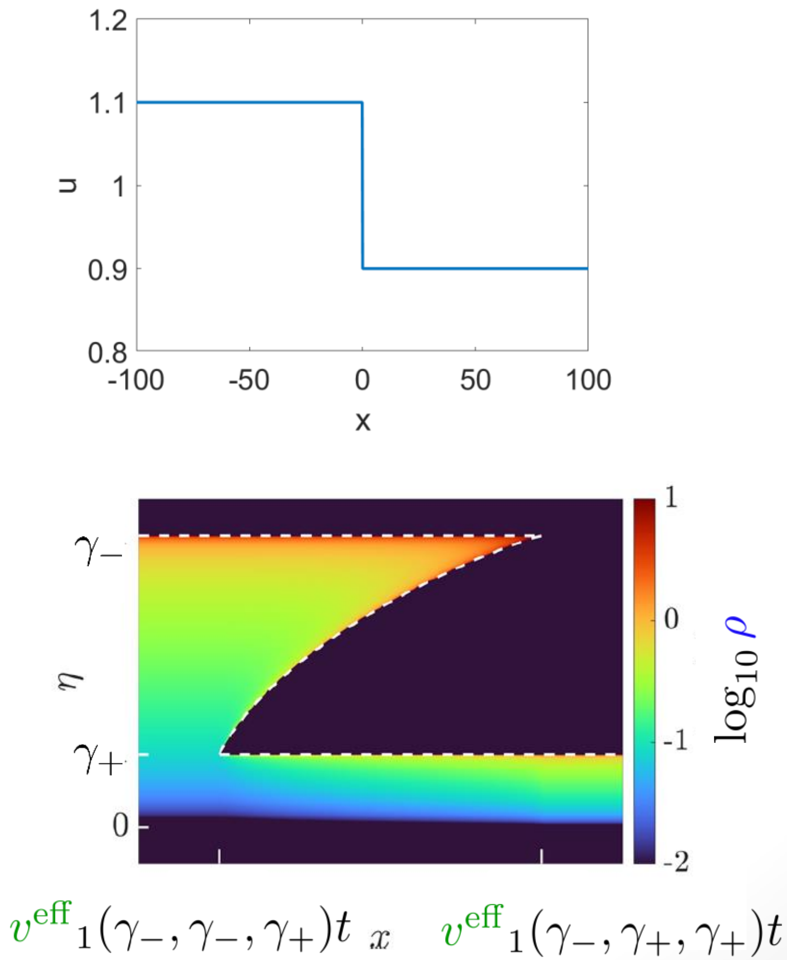

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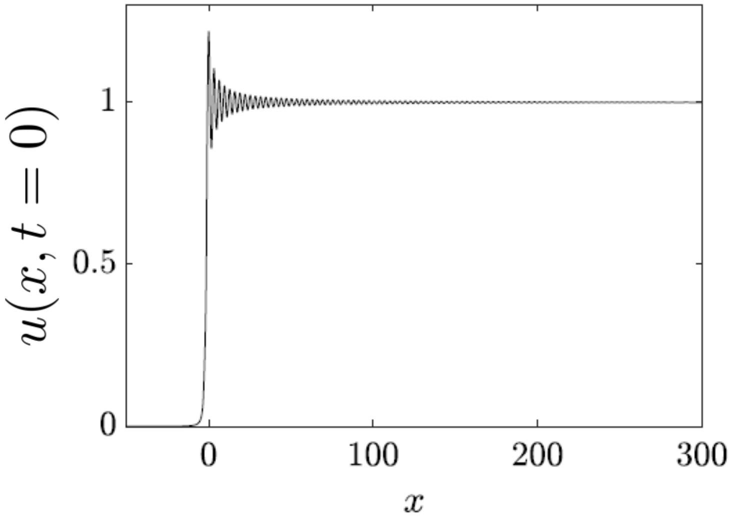

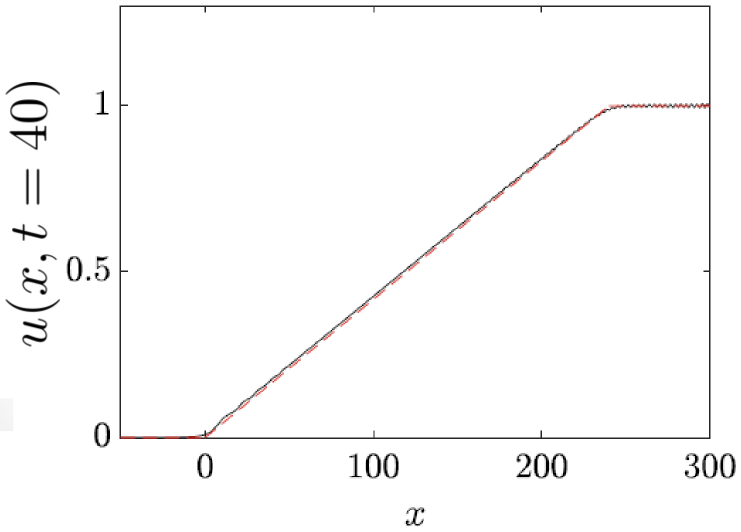

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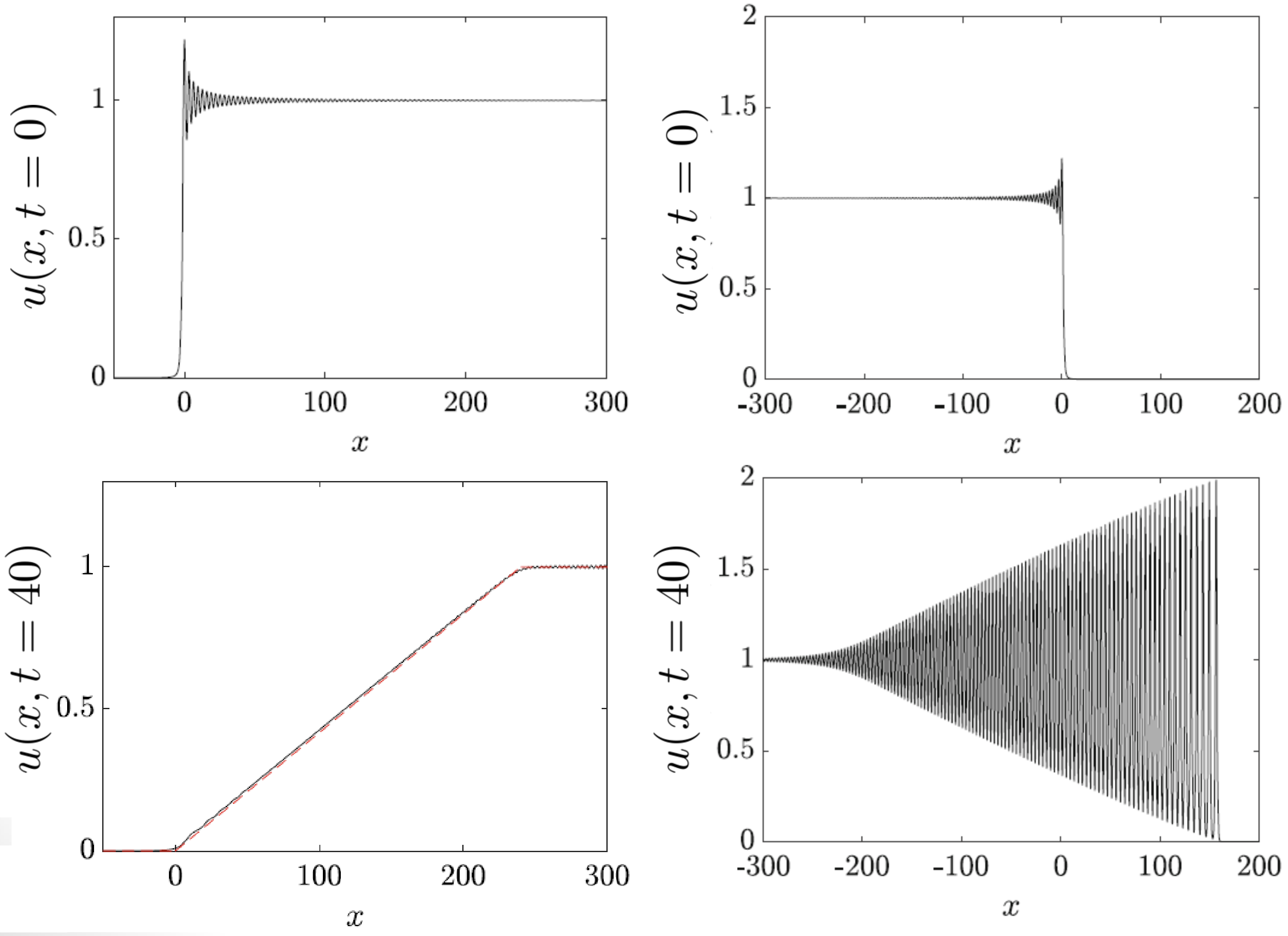

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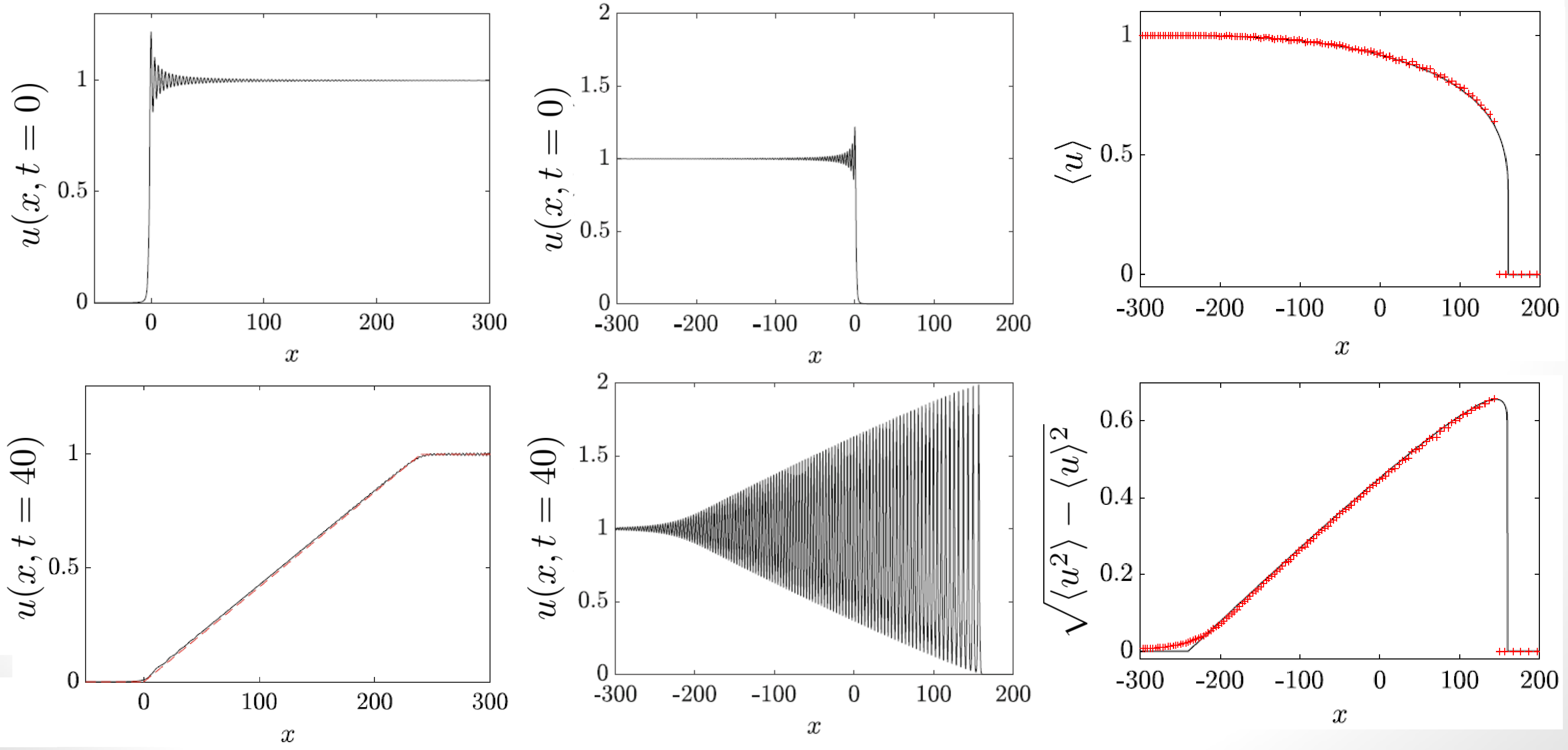

Illustration: condensate Riemann problem

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- Riemann problem, collision of a genus 0 with a genus 1 condesate (field)

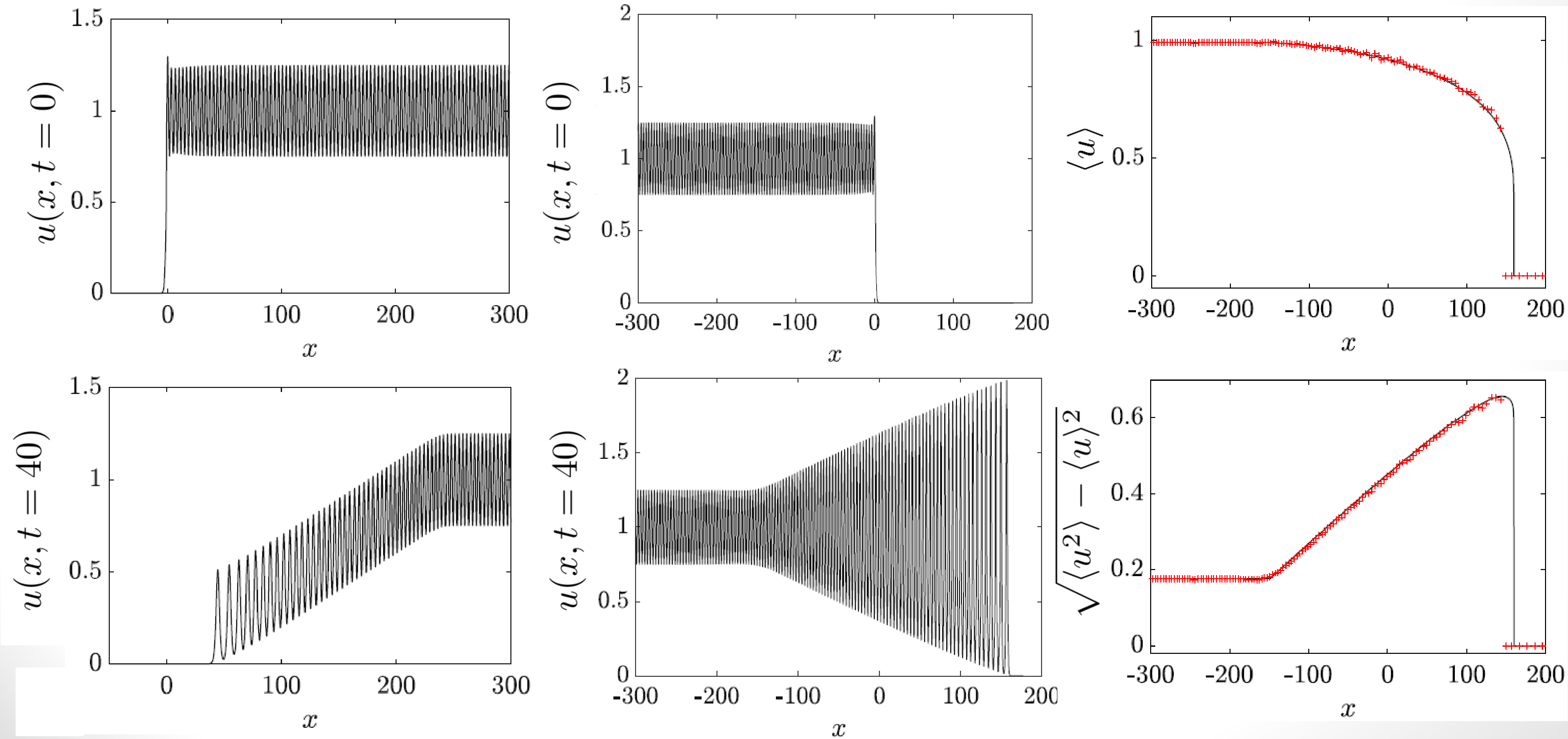

Illustration: diluted condensates

[Congy, El, Roberti, Tovbis (2023)]

- Diluted condensate of genus N: $\rho_N^{\mathrm{DC}}(\eta) \equiv \kappa \rho_N(\eta)\ , \quad 0 < \kappa < 1\ .$

$$\text{e.g.} \qquad \rho_0^{\mathrm{DC}}(\eta) = \frac{\kappa\eta}{\pi\sqrt{\gamma_0^2 - \eta^2}} \quad \Rightarrow \quad \sigma_0^{\mathrm{DC}}(\eta) = \pi \frac{1-\kappa}{\kappa}\sqrt{\gamma_0^2 - \eta^2}\ .$$

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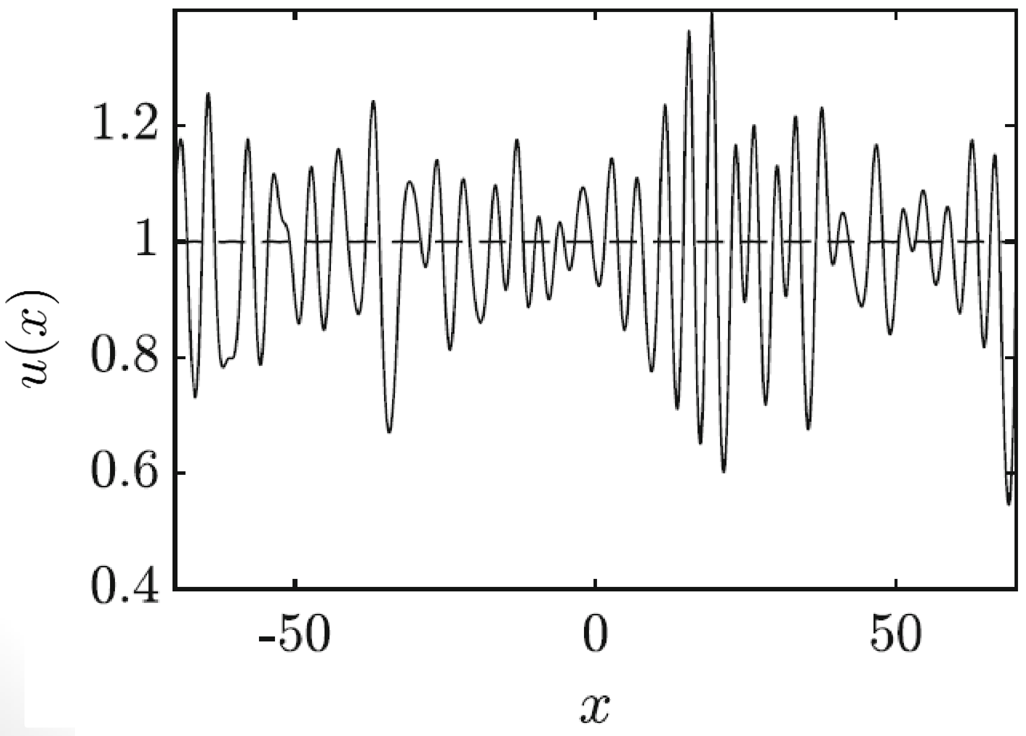


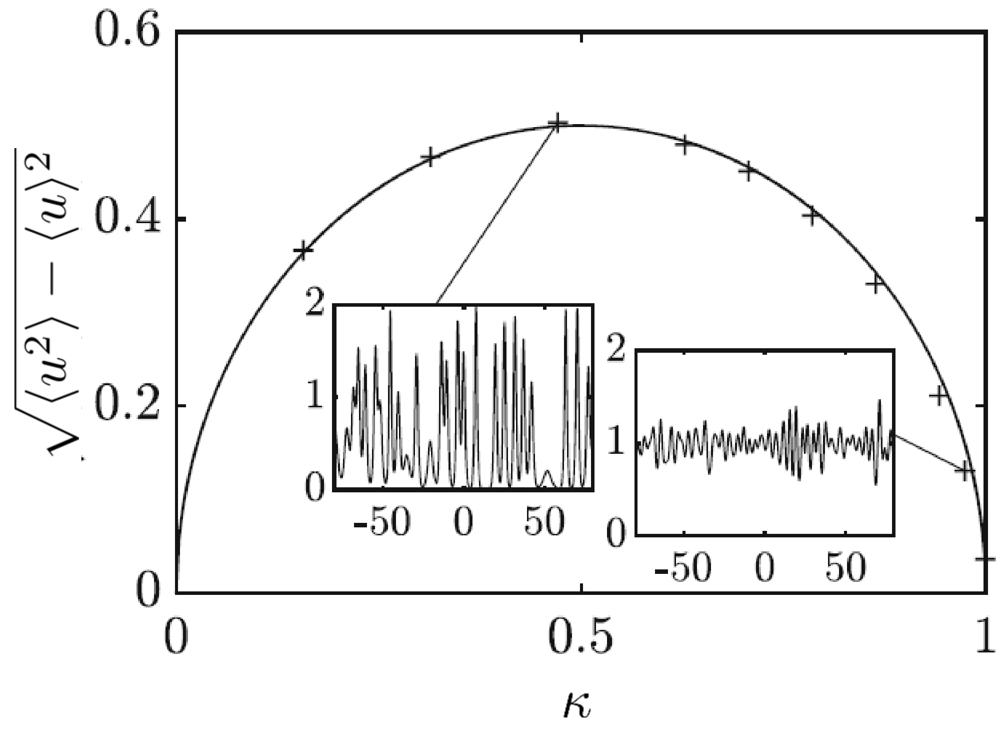
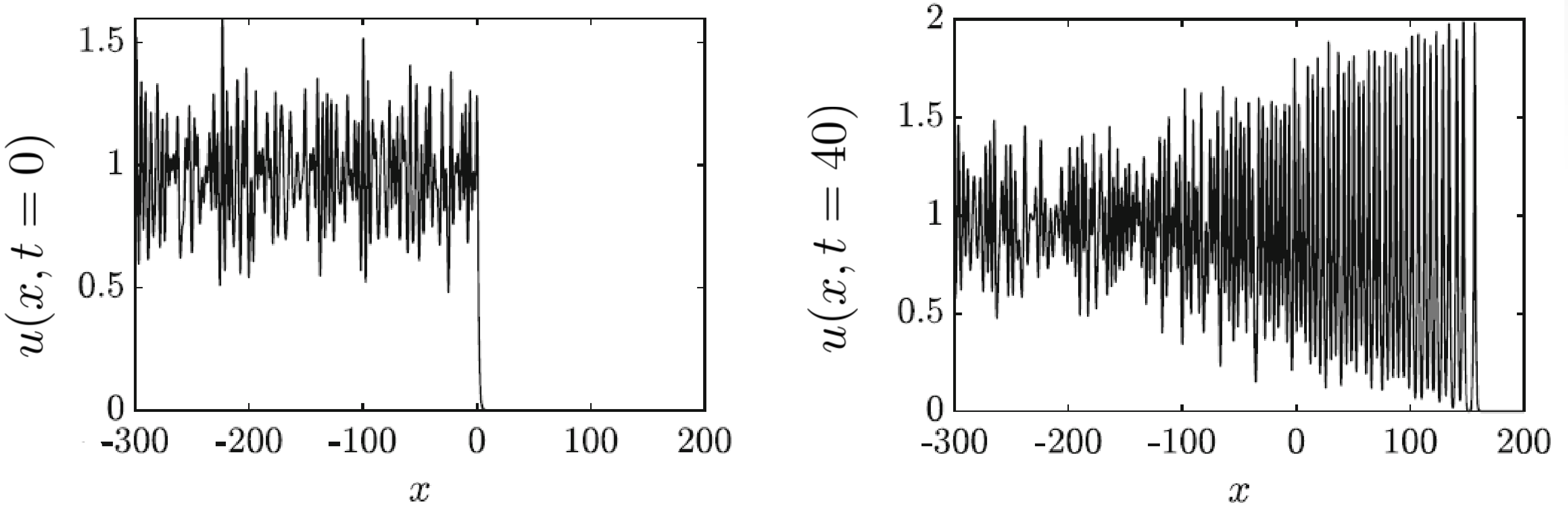


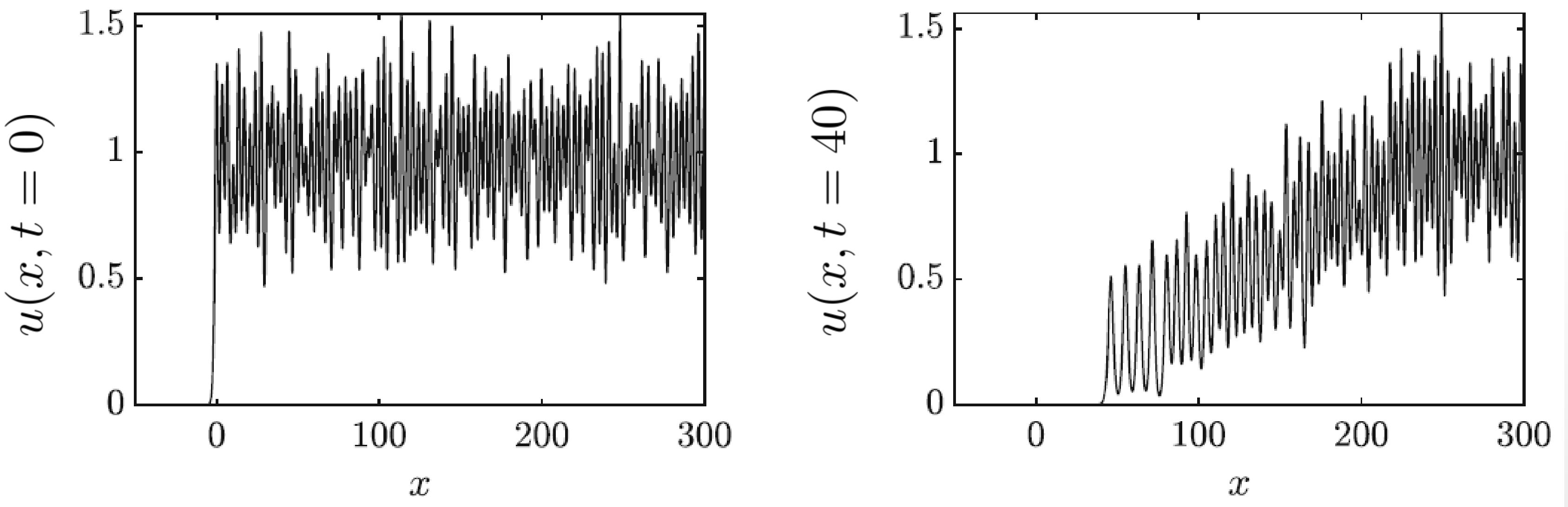
Illustration: diluted condensates Riemann problems

[Congy, El, Roberti, Tovbis (2023)]

Two genus 0
$\rightarrow$ incoherent DSW



Genus 0 and 1
$\rightarrow$ incoherent
generalised RW



Extensions to the GHD of KdV

Extensions to the GHD of KdV: correlations

[Doyon (2018)]

- Correlations between charge densities

$$\mathsf{C}_{ab} \equiv \int \mathrm{d}x \langle q_a(x) q_b(0) \rangle_c = \int_\Gamma \mathrm{d}\eta \, \rho(\eta) h_a^{\mathrm{dr}}(\eta) h_b^{\mathrm{dr}}(\eta) \ .$$

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- Correlations between charge and current densities

$$\mathsf{B}_{ab} \equiv \int \mathrm{d}x \langle j_a(x) q_b(0) \rangle_c = \int_\Gamma \mathrm{d}\eta\, \rho(\eta) v^{\mathrm{eff}}(\eta) h_a^{\mathrm{dr}}(\eta) h_b^{\mathrm{dr}}(\eta) \ .$$

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$$\mathsf{D}_{ab} \equiv \int \mathrm{d}x \langle j_a(x) j_b(0) \rangle_c = \int_\Gamma \mathrm{d}\eta\, \rho(\eta) (v^{\mathrm{eff}}(\eta))^2 h_a^{\mathrm{dr}}(\eta) h_b^{\mathrm{dr}}(\eta) \ .$$

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- Two-point correlations $(t \gg \lambda \to \infty)$

$$\frac{1}{2\lambda} \int_{-\lambda}^{\lambda} \mathrm{d}x \langle q_a(x,t) q_b(0,0)\rangle_c = \int_\Gamma \mathrm{d}\eta\, \rho(\eta) \delta\left(x - v^{\mathrm{eff}}(\eta) t\right) h_a^{\mathrm{dr}}(\eta) h_b^{\mathrm{dr}}(\eta)$$

$$= t^{-1} \sum_{\eta:\ v^{\mathrm{eff}}(\eta) = x/t} \frac{\rho(\eta) h_a^{\mathrm{dr}}(\eta) h_b^{\mathrm{dr}}(\eta))}{|\partial_\eta v^{\mathrm{eff}}(\eta)|} \ .$$

Extensions to the GHD of KdV: hydrodynamic expansion

- Euler GHD equations

$$\partial_t \rho(\eta;x,t) + \partial_x \left[v^{\text{eff}}(\eta;x,t)\rho(\eta;x,t) \right] = 0 \ ,$$

$$v^{\text{eff}}(\eta;x,t) = 4\eta^2 + \frac{1}{\eta}\int_\Gamma \log\left|\frac{\eta+\mu}{\eta-\mu}\right| \rho(\eta;x,t)[v^{\text{eff}}(\eta;x,t) - v^{\text{eff}}(\eta;x,t)]\mathrm{d}\mu \ .$$

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- "Navier-Stokes" GHD equations

$$\partial_t \rho(\eta; x, t) + \partial_x \left[v^{\text{eff}}(\eta; x, t) \rho(\eta; x, t) \right] = \frac{1}{2} \partial_x \left[\int \mathrm{d}\mu \mathcal{D}(\eta, \mu; x, t) \partial_x \rho(\mu; x, t) \right] \ .$$

[De Nardis, Bernard, Doyon (2018)]

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- "Navier-Stokes" GHD equations

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[De Nardis, Doyon (2023)]

Extensions to the GHD of KdV: breaking integrability

- Breaking integrability: KdV with (weakly) inhomogeneous coupling

$$u_t + \left[c(x)(3u^2 + u_{xx})\right]_x = 0 \ .$$

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[Doyon, Yoshimura (2017)] $\Downarrow$ *[Bastinello, Alba, Caux (2019)]*

$$\partial_t \rho + \partial_x \left[v^{\text{eff}} \rho\right] + \partial_\eta \left[a^{\text{eff}} \rho\right] = 0 \ ,$$

$$a^{\text{eff}}(\eta; x, t) = -\eta^3 \partial_x c(x) + \frac{1}{\eta} \int_\Gamma \mathrm{d}\mu \log\left|\frac{\eta+\mu}{\eta-\mu}\right| \rho(\mu; x, t) \left[a^{\text{eff}}(\mu; x, t) - a^{\text{eff}}(\eta; x, t)\right] \ .$$

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- What about KdV-Burgers?

[Based on: Bouchoule, Doyon, Dubail (2020) ?]

$$\partial_t u + 6u\partial_x u + \partial_{xxx} u = \epsilon \partial_{xx} u \ .$$

GHD in other models: NLS

- (Focusing) NLS and its soliton solution

$$i\psi_t + \psi_{xx} + 2|\psi|^2\psi = 0 \ ,$$

$$\psi_1(x,t) = 2ib\,\mathrm{sech}\left[2b(x + 4at - x_0)\right]\exp\left[-2i\left(ax + 2(a^2 - b^2) - \phi_0\right)\right] \ .$$

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- Soliton gas approach: thermodyanmic limit of finite gap solutions

[El, Kamchatnov (2005)]
[El, Tovbis (2020)]

$$\int_\Gamma \log\left|\frac{\mu - \bar\eta}{\mu - \eta}\right| \rho(\mu)|\mathrm{d}\mu| \ + \ \sigma(\mu)\rho(\mu) = \mathrm{Im}(\eta) \ ,$$

$$\int_\Gamma \log\left|\frac{\mu - \bar\eta}{\mu - \eta}\right| f(\mu)|\mathrm{d}\mu| \ + \ \sigma(\mu)f(\mu) = -4\mathrm{Im}(\eta)\mathrm{Re}(\eta) \ .$$

Solitonic NDRs

$$\int_\Gamma \rho(\mu)\left[\arg\frac{\mu - \bar\eta}{\mu - \eta} - \arg\mu\right]|\mathrm{d}\mu| = \mathrm{Re}(\eta) + \tilde\rho(\eta) \ ,$$

$$\int_\Gamma f(\mu)\left[\arg\frac{\mu - \bar\eta}{\mu - \eta} - \arg\mu\right]|\mathrm{d}\mu| = -2\mathrm{Re}(\eta^2) + \tilde f(\eta) \ .$$

Carrier NDRs

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Carrier NDRs

- GHD approach:

[Koch, Caux, Bastianello (2022)]

GHD of (attractive) Lieb-Liniger $\quad \rightarrow \quad$ take the semiclassical limit.

GHD in other models: Boussinesq

- (Good) Boussinesq equation and its soliton solution

$$u_{tt} - u_{xx} = -\left[6\left(u^2\right)_{xx} + u_{xxxx}\right] \ ,$$

$$u_1(x,t) = \left(\frac{\eta}{2}\right)^2 \operatorname{sech}^2\left[\frac{\eta}{2}\left(x - \epsilon t\sqrt{1-\eta^2} - x_0\right)\right] \ , \quad \epsilon = \pm 1 \ .$$

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- GHD approach: dressing operation

[Bonnemain, Doyon (2024)]

$$\begin{cases} h^{\mathrm{l,dr}}(\eta) = h^{\mathrm{l}}(\eta) + \displaystyle\int_{\Gamma_{\mathrm{l}}} \frac{\mathrm{d}\mu}{2\pi}\, \varphi_{\mathrm{O}}(\eta,\mu) n_{\mathrm{l}}(\mu) h^{\mathrm{l,dr}}(\mu) + \int_{\Gamma_{\mathrm{r}}} \frac{\mathrm{d}\mu}{2\pi} \varphi_{\mathrm{H}}(\eta,\mu) n_{\mathrm{r}}(\mu) h^{\mathrm{r,dr}}(\mu) \\ h^{\mathrm{r,dr}}(\eta) = h^{\mathrm{r}}(\eta) + \displaystyle\int_{\Gamma_{\mathrm{r}}} \frac{\mathrm{d}\mu}{2\pi}\, \varphi_{\mathrm{O}}(\eta,\mu) n_{\mathrm{r}}(\mu) h^{\mathrm{r,dr}}(\mu) + \int_{\Gamma_{\mathrm{l}}} \frac{\mathrm{d}\mu}{2\pi} \varphi_{\mathrm{H}}(\eta,\mu) n_{\mathrm{l}}(\mu) h^{\mathrm{l,dr}}(\mu) \end{cases}$$

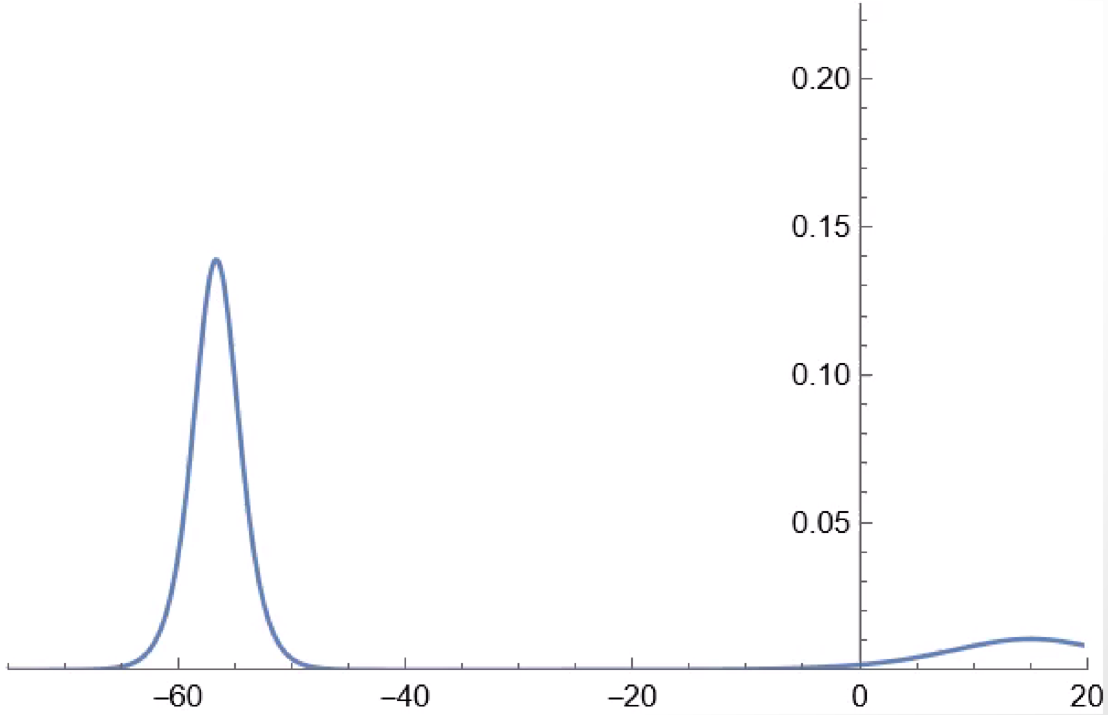
GHD in other models: Boussinesq

- (Good) Boussinesq equation and its soliton solution $u_{tt} - u_{xx} = -\left[6\left(u^2\right)_{xx} + u_{xxxx}\right]$.
- GHD approach: dressing operation *[Bonnemain, Doyon (2024)]*

$$h^{\mathrm{l/r,dr}}(\eta) = h^{\mathrm{l/r}}(\eta) + \int_{\Gamma_{\mathrm{l/r}}} \frac{\mathrm{d}\mu}{2\pi}\, \varphi_{\mathrm{O}}(\eta,\mu) n_{\mathrm{l}}(\mu) h^{\mathrm{l/r,dr}}(\mu) + \int_{\Gamma_{\mathrm{r/l}}} \frac{\mathrm{d}\mu}{2\pi} \varphi_{\mathrm{H}}(\eta,\mu) n_{\mathrm{r}}(\mu) h^{\mathrm{r/l,dr}}(\mu)$$

- Boussinesq peculiarity: soliton merging

Knowledge of the asymptotic train of solitons is not enough to construct the thermodynamics

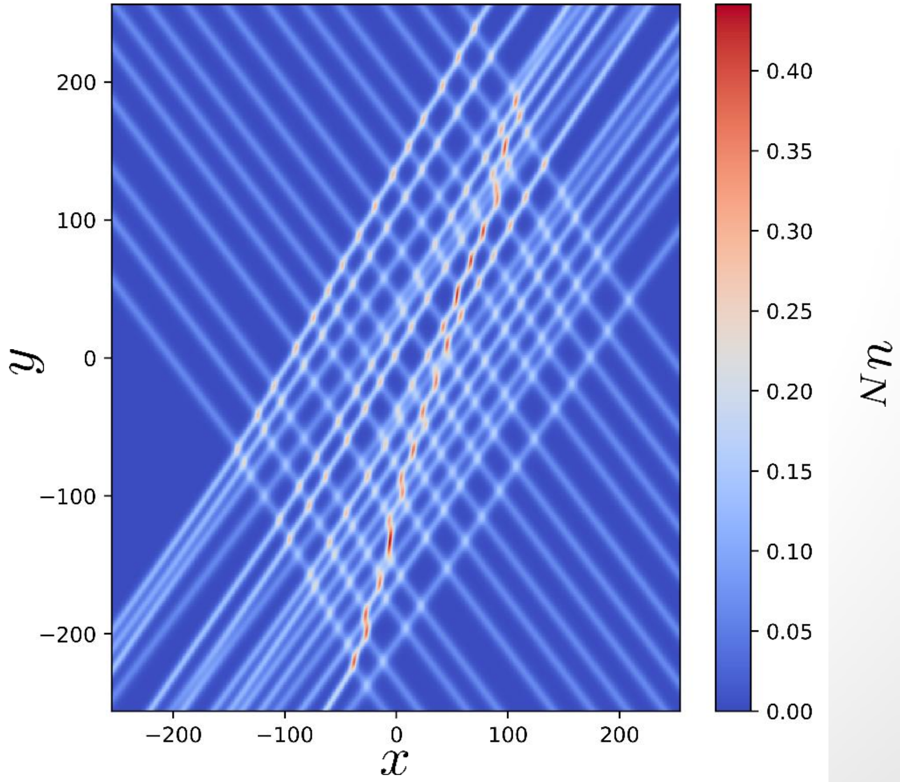


GHD in other models: Kadomtsev-Petviashvili

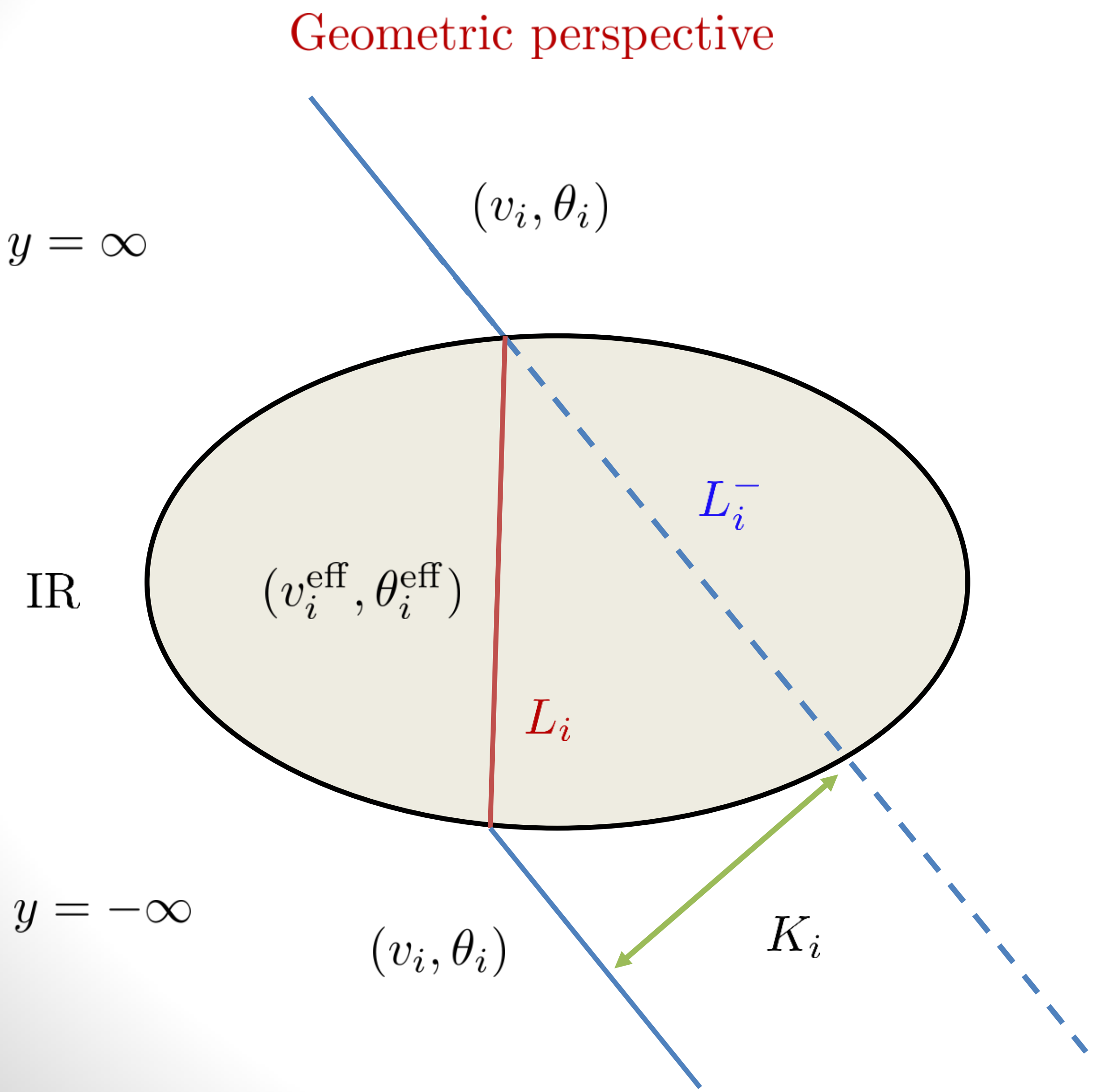
- KP: "KdV in $(2+1)$d" with line solitons

$$\left[u_t + 6(u^2)_x + u_{xxx}\right]_x + \alpha u_{yy} = 0 \ ,$$

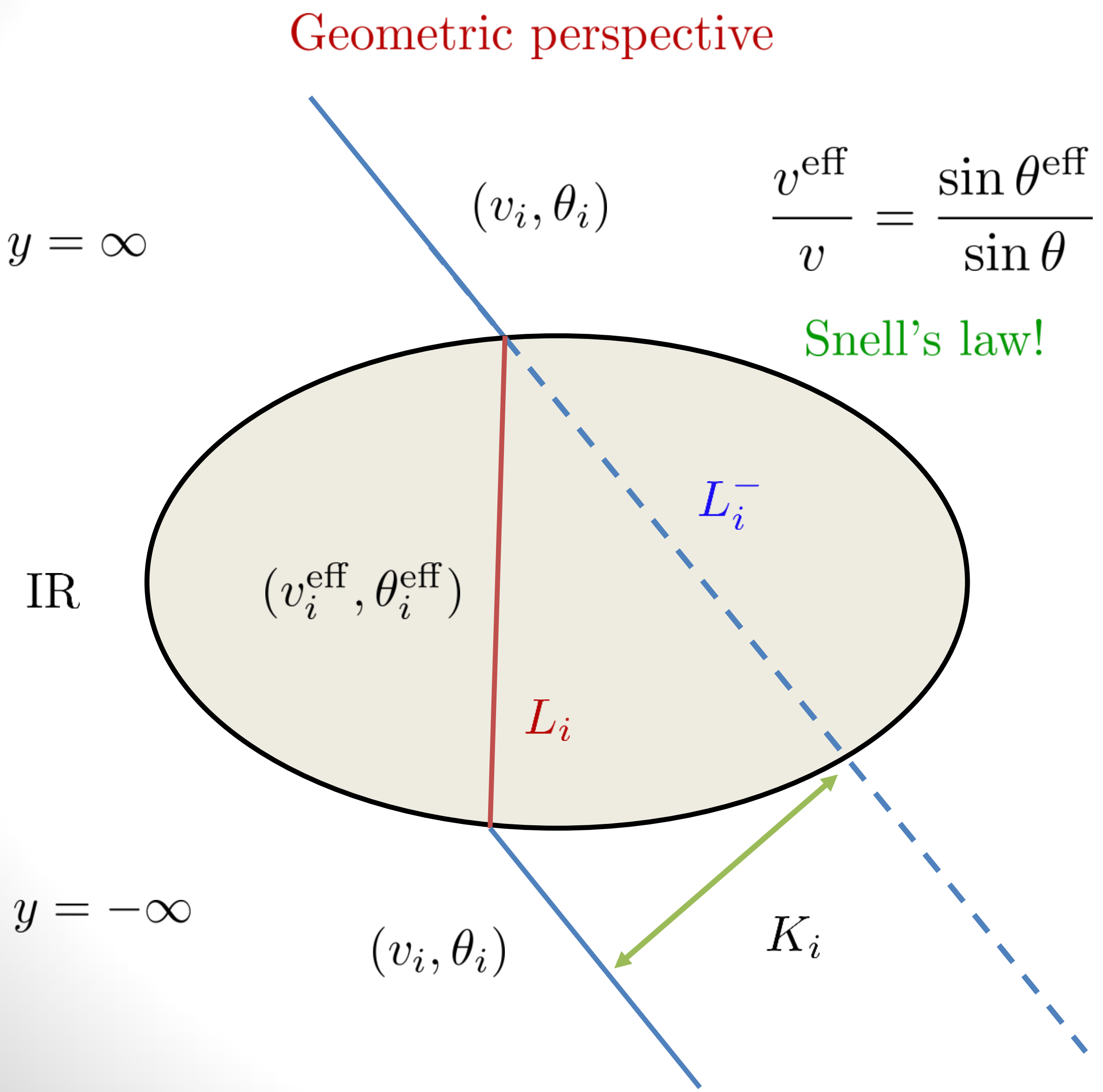
$$u_1(x,y,t) = 2a^2 \mathrm{sech}^2[a(x - cy - vt) + \phi_0] \ .$$



GHD in other models: Kadomtsev-Petviashvili

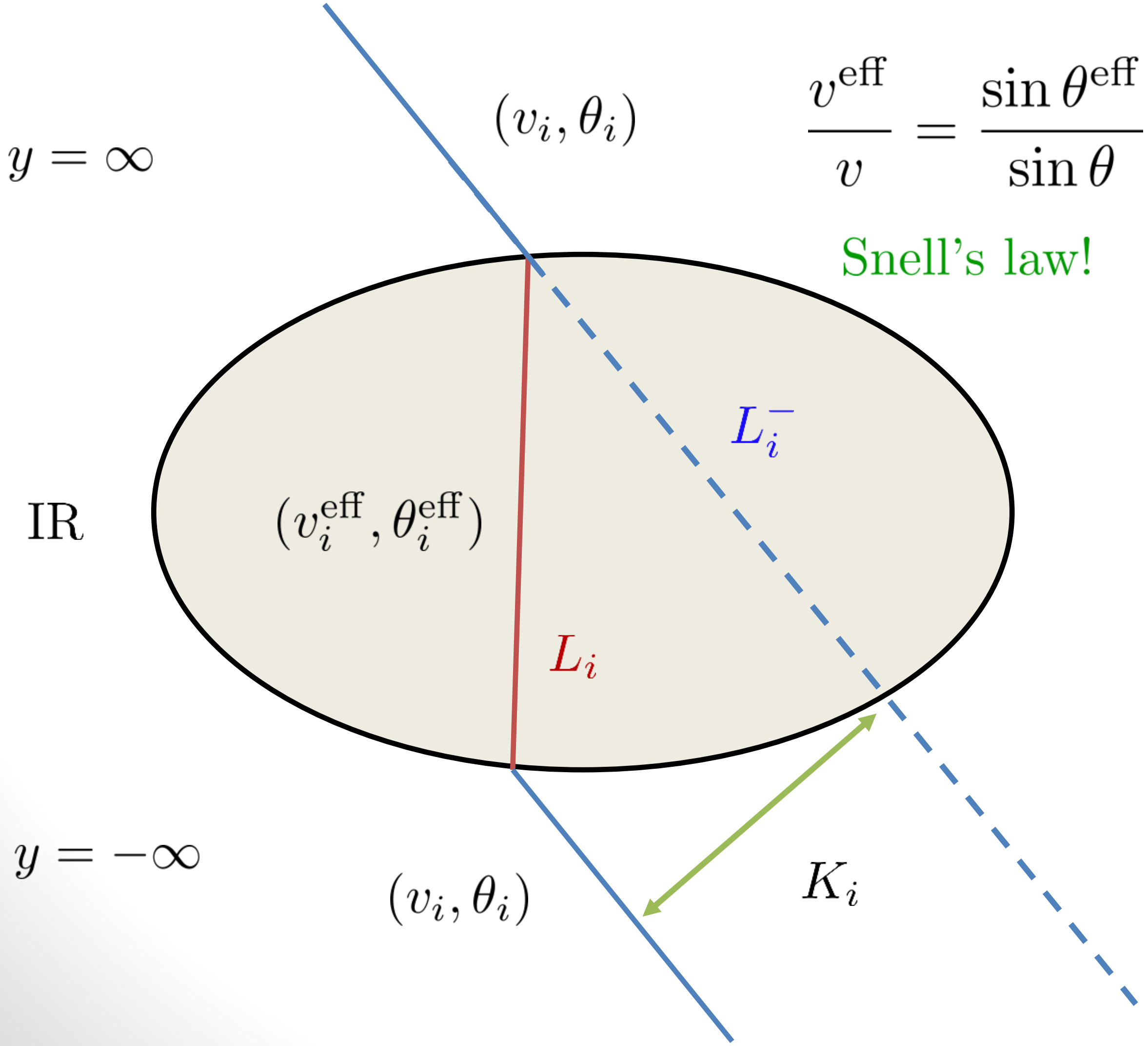


GHD in other models: Kadomtsev-Petviashvili



GHD in other models: Kadomtsev-Petviashvili

Geometric perspective

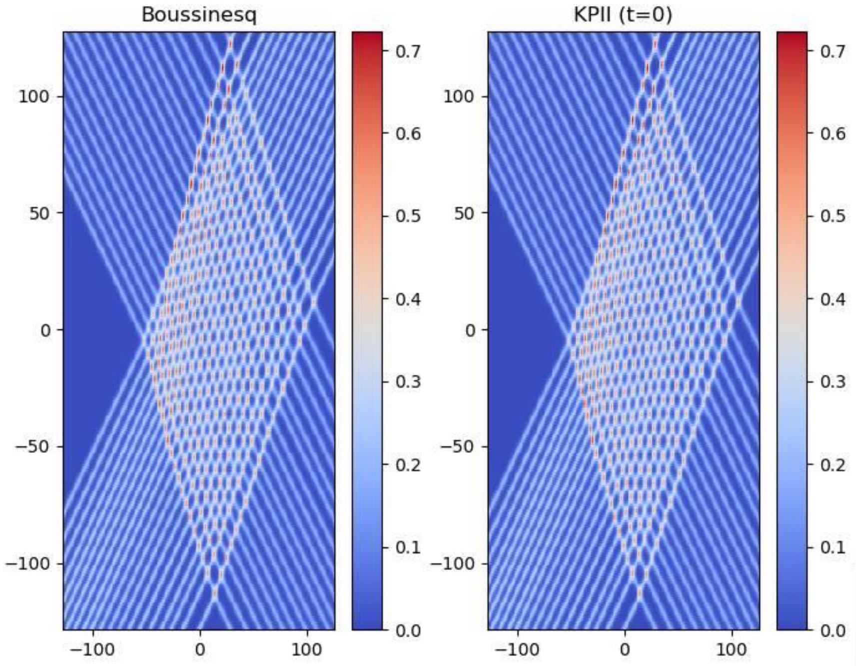


$$\frac{v^{\mathrm{eff}}}{v} = \frac{\sin\theta^{\mathrm{eff}}}{\sin\theta}$$

Snell's law!

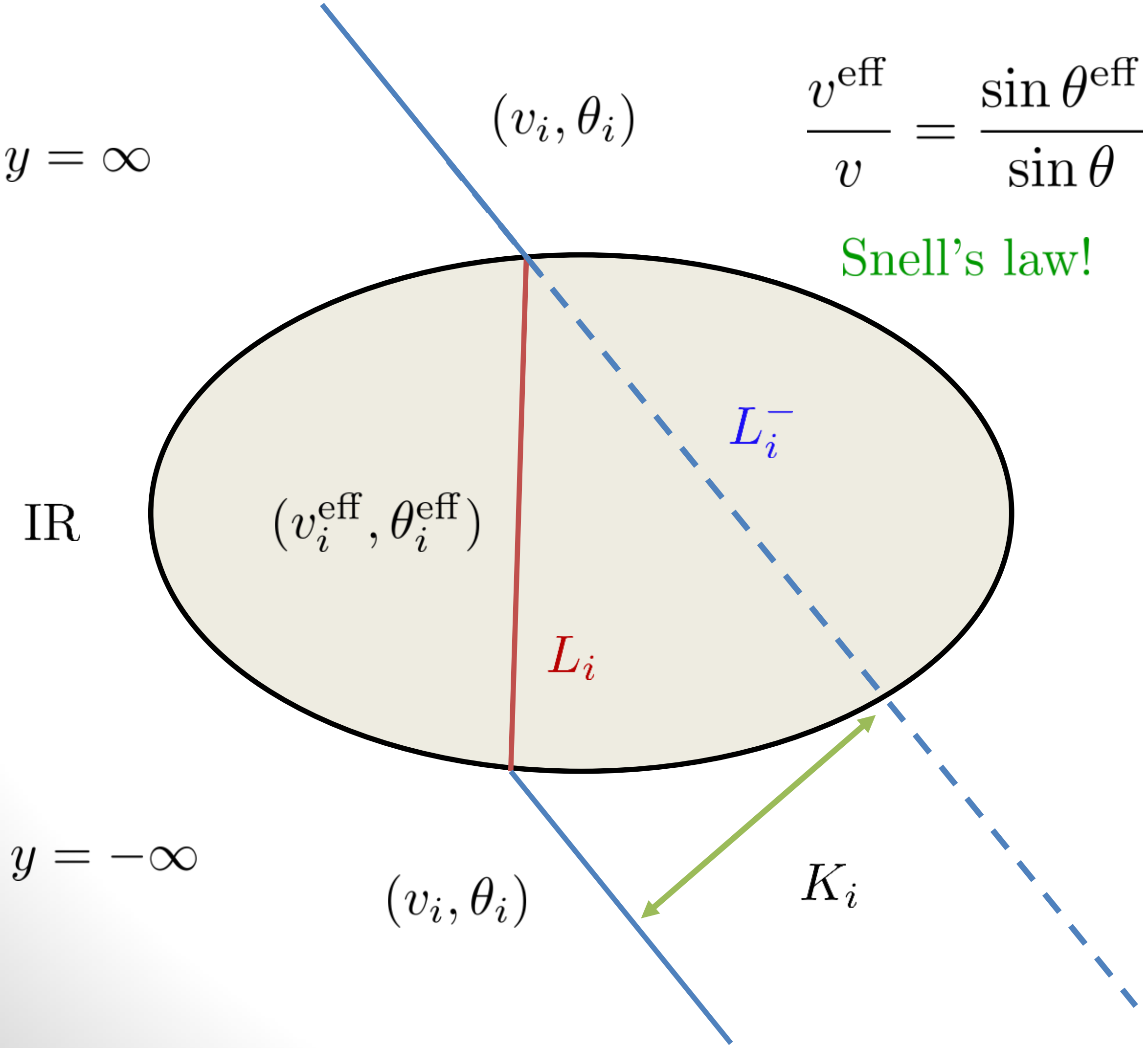
Analogy with $(1+1)$d models

- Boussinesq: stationary reduction of KP.



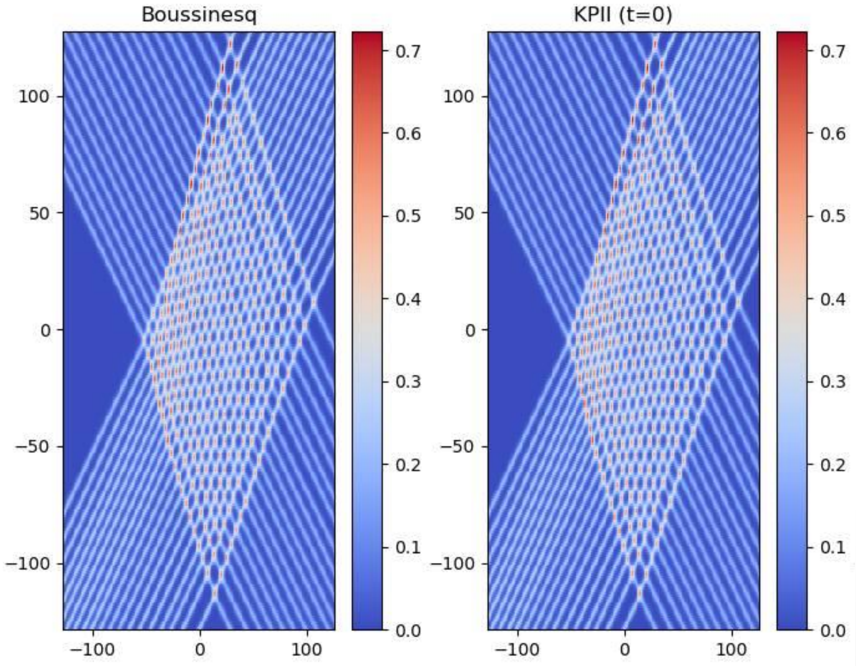
GHD in other models: Kadomtsev-Petviashvili

Geometric perspective



Analogy with $(1+1)$d models

- Boussinesq: stationary reduction of KP.



$$\Rightarrow \quad (v_{\text{2D}}^{\text{eff}})^2 \left[1 + (v_{\text{1D}}^{\text{eff}})^2\right] = v_{\text{2D}}^2 \left[1 + v_{\text{1D}}^2\right]$$

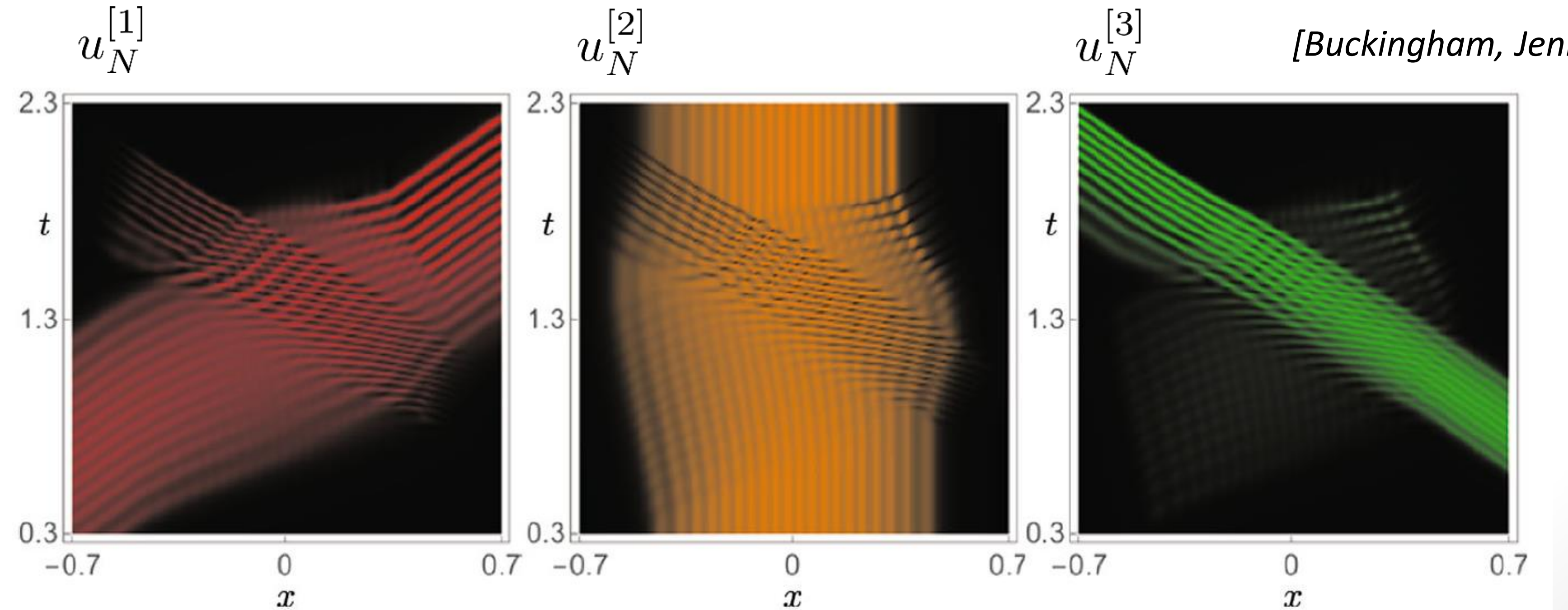
GHD in other models: 3-wave equation

- "Weakly integrable" model: the 3-wave equation

$$\epsilon\left(u_t^{[j]} + c^{[j]} u_x^{[j]}\right) = \gamma^{[j]} \bar{u}^{[k]} \bar{u}^{[l]} \ , \quad \text{with} \quad j,\ k,\ l = 1,\ 2,\ 3 \quad \text{cyclic}\ ,$$

$$u_1^{[j]}(x,y,t) = \gamma^{[j]} e^{i\left(\phi_0{}^{[j]} - a\Delta^{[j]}\xi^{[j]}/\epsilon\right)} b\Delta^{[j]} \sqrt{\Delta^{[k]}\Delta^{[l]}} \operatorname{sech}(b\Delta^{[j]}\xi^{[j]}/\epsilon)\ ,$$

$$\xi^{[j]} = x - c^{[j]} t - x_0{}^{[j]}\ .$$



[Buckingham, Jenkins, Miller (2017)]

GHD in other models: 3-wave equation

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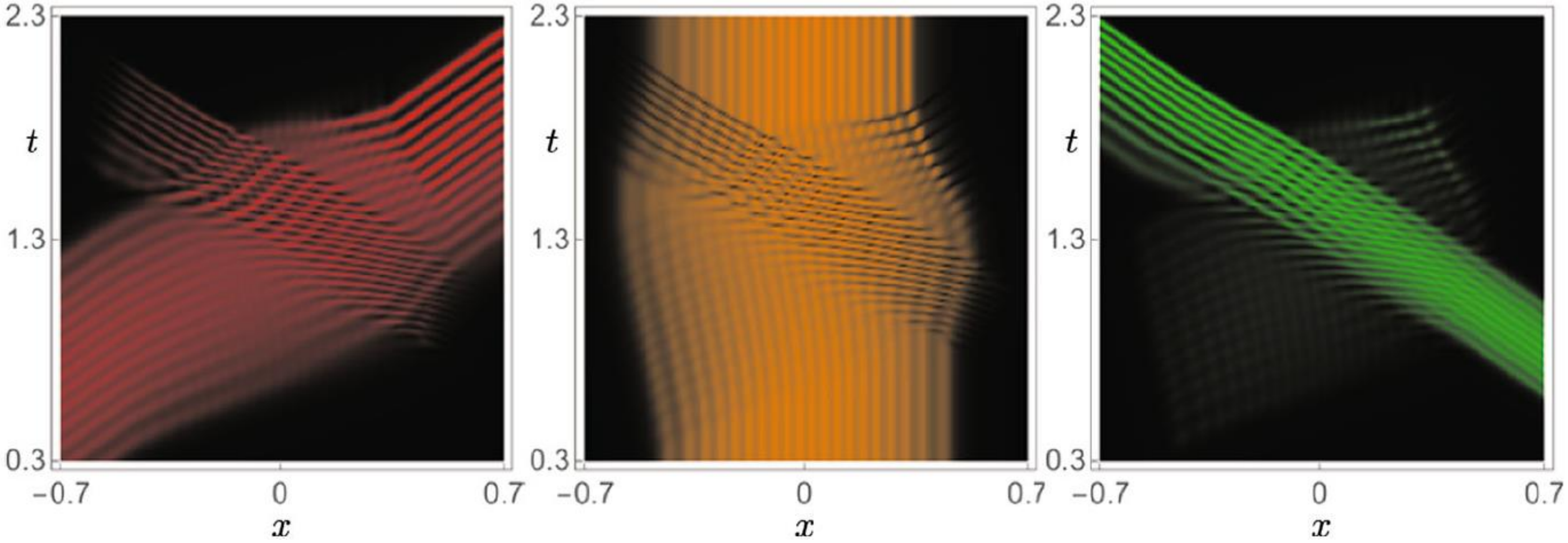

[Buckingham, Jenkins, Miller (2017)]

- Peculiarities of the 3-wave equation

Set of conserved charges is not complete

Scattering not 2-body reducible

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