Introduction to Generalised Hydrodynamics in integrable field theories

Disordered Systems Advanced Lectures Series
3nd lecture

Thibault Bonnemain, 29th January 2024

Recap of the previous lectures

- KdV: integrable, nonlinear, dispersive PDE

$$\partial_t u + 6u\partial_x u + \partial_x^3 u = 0 \ .$$

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- KdV is an Hamiltonian system

$$\partial_t u = \{\mathcal{H}; u\} = \partial_x \frac{\delta \mathcal{H}}{\delta u(x)} \ ,$$

$$\text{with} \quad \{\mathcal{F}; \mathcal{G}\} = \int_{\mathbb{R}} \mathrm{d}x \ \frac{\delta \mathcal{F}}{\delta u(x)} \partial_x \frac{\delta \mathcal{G}}{\delta u(x)} \ , \quad \text{and} \quad \mathcal{H} = \int_{\mathbb{R}} \mathrm{d}x \ \frac{u_x^2}{2} - u^3 \ .$$

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- Infinite set of conservation laws

Time conserved "charges"

$$Q_n = \int \mathrm{d}x \ q_n(x,t) \ , \quad \text{and} \quad J_n = \int \mathrm{d}t \ j_n(x,t) \ ,$$

Space conserved "currents"

$$\boxed{\partial_t q_n + \partial_x j_n = 0 \ .}$$

Recap of the previous lectures

- Solvable via IST: KdV is the compatibility condition for a linear problem

$$\mathcal{L}\phi = \lambda\phi \ , \qquad \phi_t = \mathcal{M}\phi \ ,$$

$$\mathcal{L} = -\partial_{xx} - u(x,t) \ , \quad \mathcal{M} = u_x + [4\lambda - 2u(x,t)]\partial_x \ , \quad \dot{\mathcal{L}} = [\mathcal{M};\mathcal{L}] \ .$$

Lax pair

Lax equation ⇔ KdV

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- Finite gap theory: associates (multi-)periodic solution of KdV with a particular band spectrum of $\mathcal{L}$

$$\lambda \in [\lambda_1, \lambda_2] \cup \cdots \cup [\lambda_{2N+1}, +\infty[\ .$$

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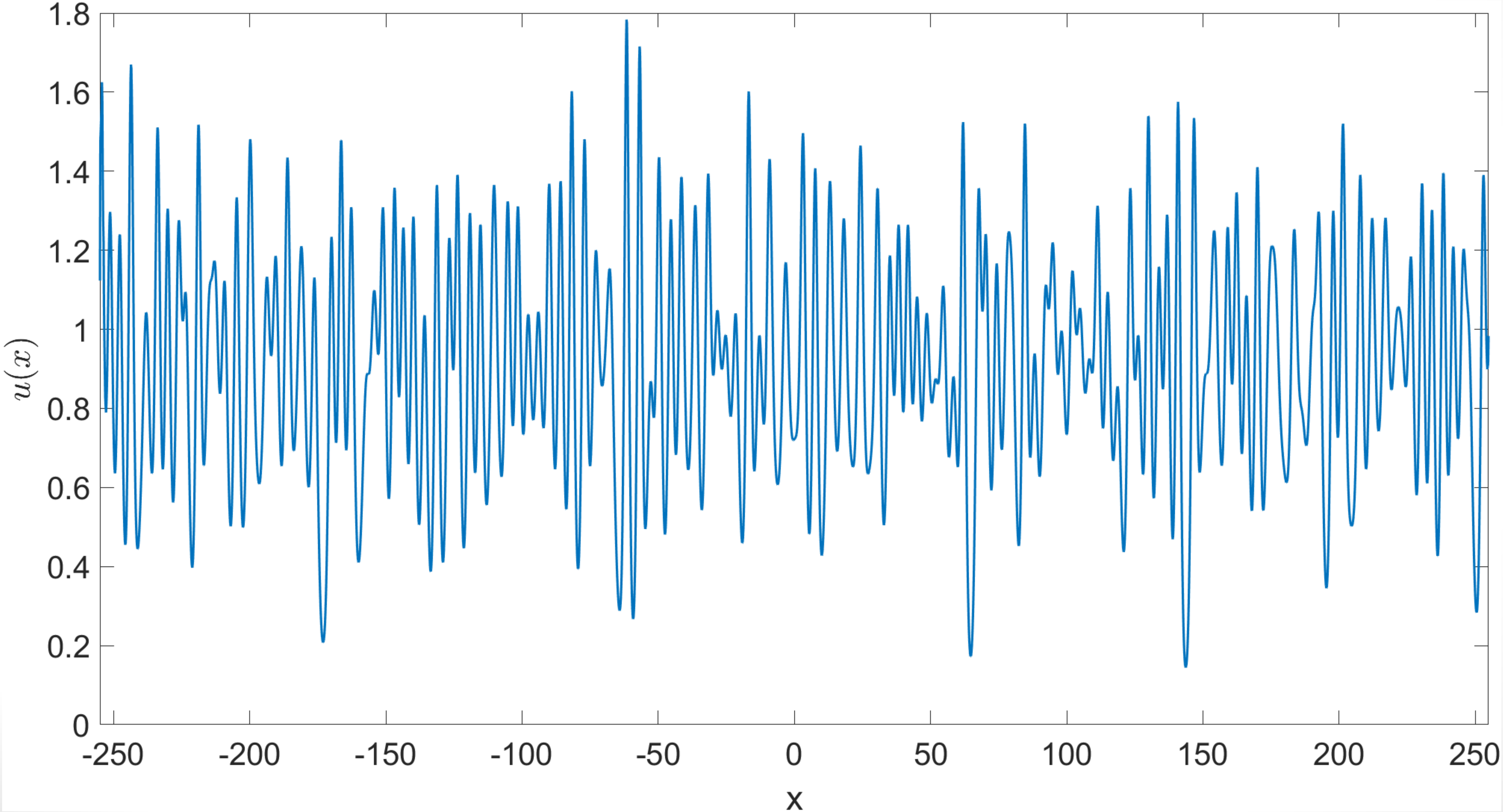
- $N-$soliton solutions: all N band shrink to points

$$u_N(x,t) \approx \sum_{i=1}^{N} 2\eta_i^2 \mathrm{sech}^2 \left[\eta_i \left(x - 4\eta_i^2 t - x_i^{\pm}\right)\right] \quad \text{as} \quad t \to \pm\infty.$$

Soliton gas: basic idea and motivations



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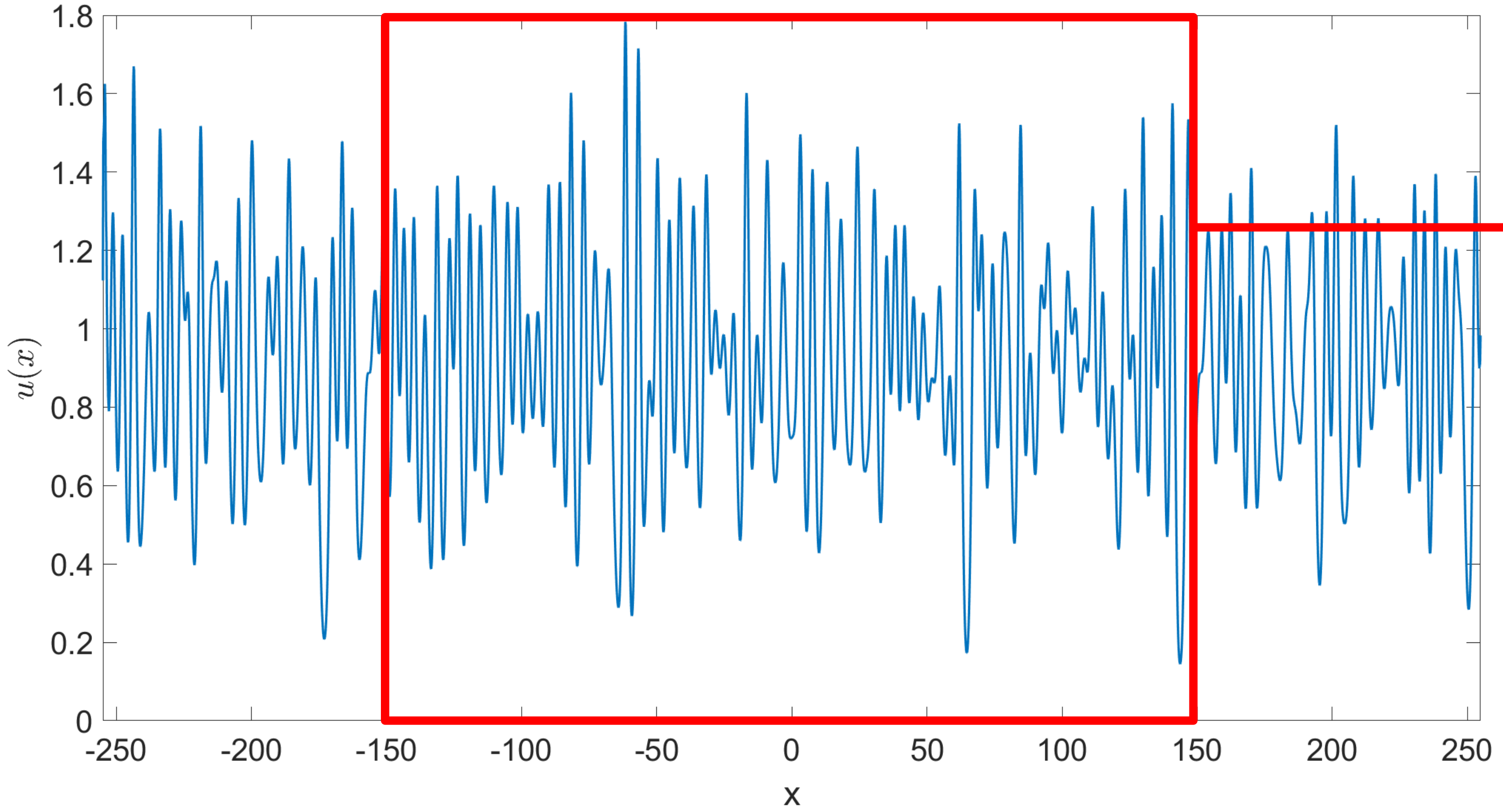


Fluid cell of size L containing N solitons

$$u_L(x,0) = \begin{cases} u(x) \ , \ |x| < \frac{L}{2} \\ 0 \ , \ \text{otherwise} \end{cases}$$

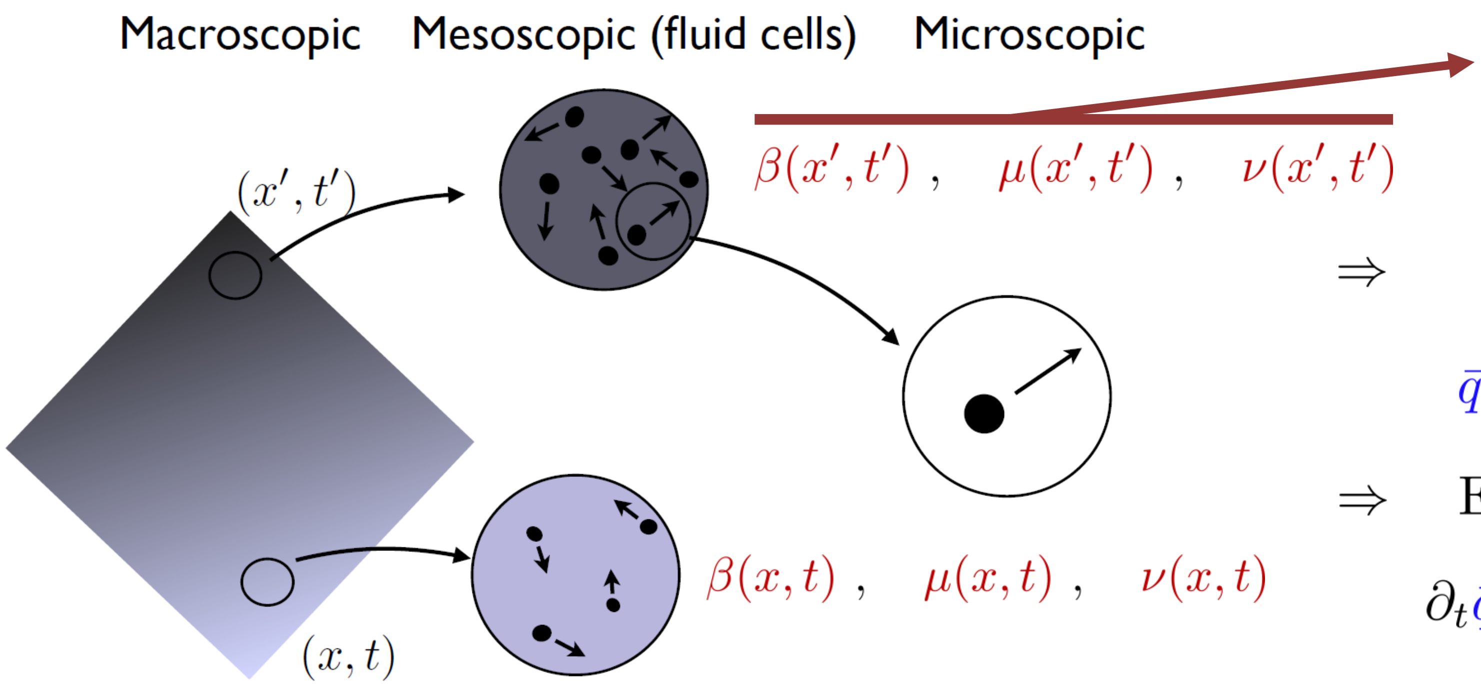
Asymptotically: $u_L(x,t) \approx \sum_{i=1}^{N} 2\eta_i^2 \text{sech}^2 \left[\eta_i \left(x - 4\eta_i^2 t - x_i^{\pm}\right)\right]$ as $t \to \pm\infty$.

Large scale dynamics of inhomogeneous soliton gas?

- Non-integrable systems thermalise to Gibbs ensembles (GE)

$$\mu_{\mathrm{GE}} = \frac{1}{Z_{\mathrm{GE}}} \sum_{N=0}^{\infty} \exp\left[-\beta(E - \mu N - \nu P)\right] \frac{1}{N!} \mathrm{d}^N \mathbf{x} \mathrm{d}^N \mathbf{p} \ .$$

- Hydrodynamic principle: separation of scales and propagation of local GE



$\Rightarrow$ Local GE averages

$\bar{q}_m(x,t)$ and $\bar{j}_m(x,t)$

$\Rightarrow$ Euler hydrodynamics

$$\partial_t \bar{q}_m(x,t) + \partial_x \bar{j}_m(x,t) = 0$$

[Doyon: Lecture Notes (2020)]

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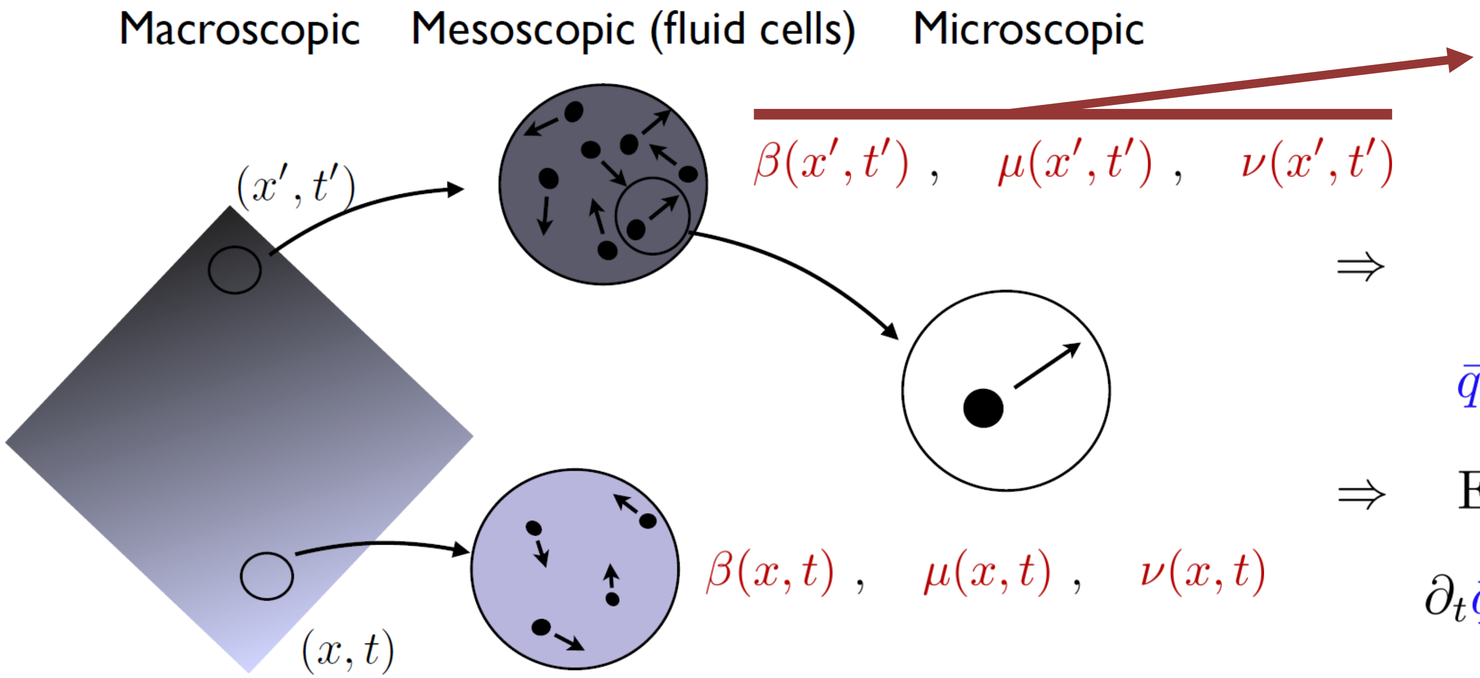
Lesson from FPUT:
Integrability prevents thermalisation!

- Hydrodynamic principle: separation of scales and propagation of local GE



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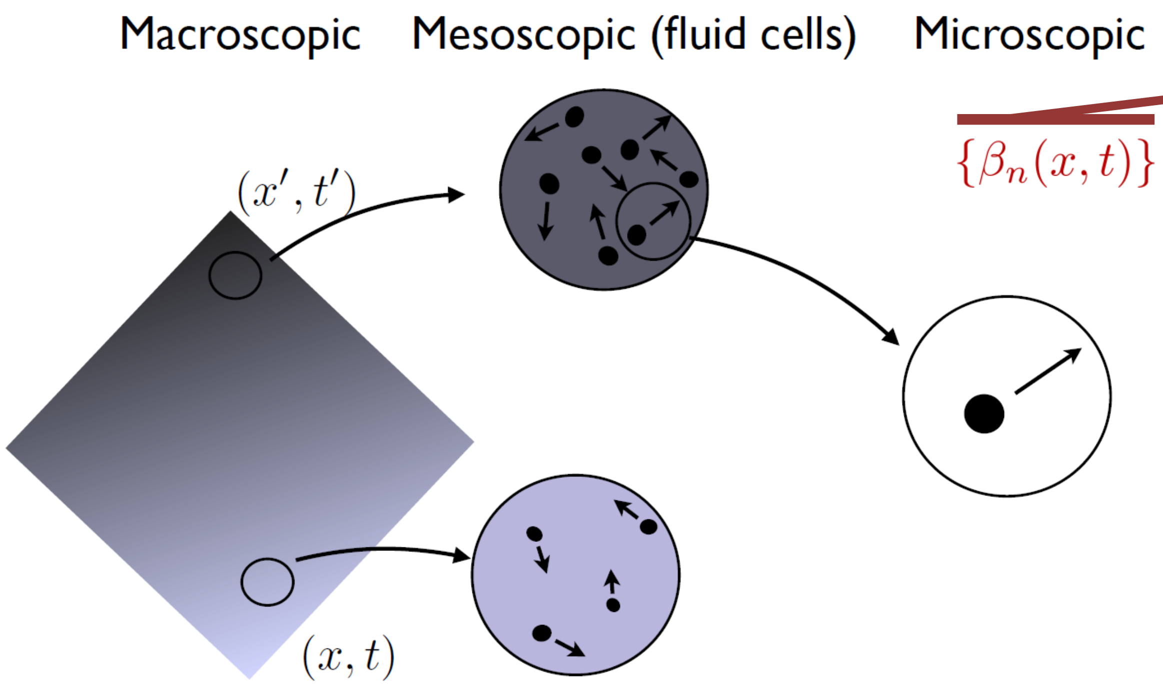
- Non-integrable systems thermalise to Generalised Gibbs ensembles (GGE)

[Rigol et al. (2007)]

$$\mu_{\mathrm{GGE}} = \frac{1}{Z_{\mathrm{GGE}}} \sum_{N=0}^{\infty} \exp\left[-\sum_{k=0}^{\infty} \beta_k Q_k\right] \frac{1}{N!} \mathrm{d}^N \mathbf{x} \mathrm{d}^N \mathbf{p} \ .$$

- Hydrodynamic principle: separation of scales and propagation of local GGE



[Doyon: Lecture Notes (2020)]

$\{\beta_n(x,t)\}$ → Slowly varying

$\Rightarrow$ Local GGE averages

$\bar{q}_m(x,t)$ and $\bar{j}_m(x,t)$

$\Rightarrow$ Generalised hydrodynamics

???

GHD in a nutshell

	Homogeneous - Stationary		Inhomogeneous - Dynamical
Finite number of conserved quantities	Statistical Thermodynamics	⇒	Hydrodynamics
	⇓		⇓
Infinite number of conserved quantities	Generalized Gibbs Ensembles	⇒	Generalized Hydrodynamics

Outline of the lectures

I. **Elements of Hydrodynamics**

II. **Integrable field theories**

III. **Soliton gas and Generalised Hydrodynamics**

1) Mise en bouche: Zakharov's rarefied gas (1971).
2) Thermodynamics of the KdV gas: quasi-particle (heurisitic) approach.
3) Soliton gas from finite gap theory: non-linear dispersion relations (sketch of derivation).
4) (Generalised) Hydrodynamics of the KdV gas.

IV. **Specific examples and potential extensions**

Zakharov's rarefied gas (1971)

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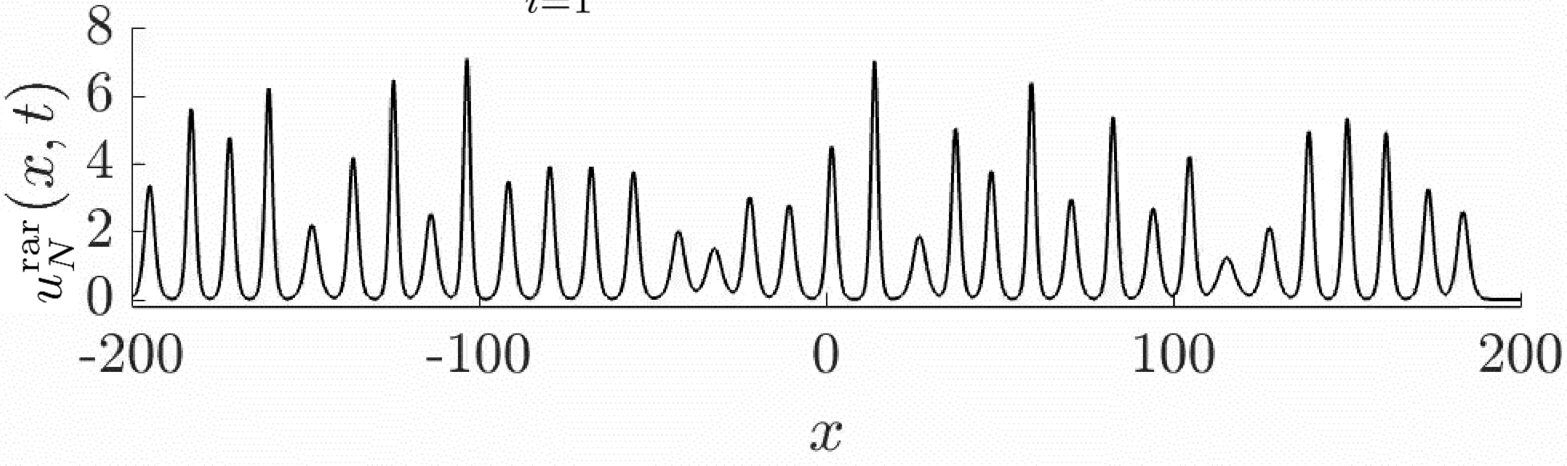
- Random solution that almost everywhere in time can be approximated by

$$u_N^{\mathrm{rar}}(x,t) \approx \sum_{i=1}^{N} 2\eta_i^2 \mathrm{sech}^2\left[\eta_i\left(x - 4\eta_i^2 t - x_i\right)\right] \;.$$



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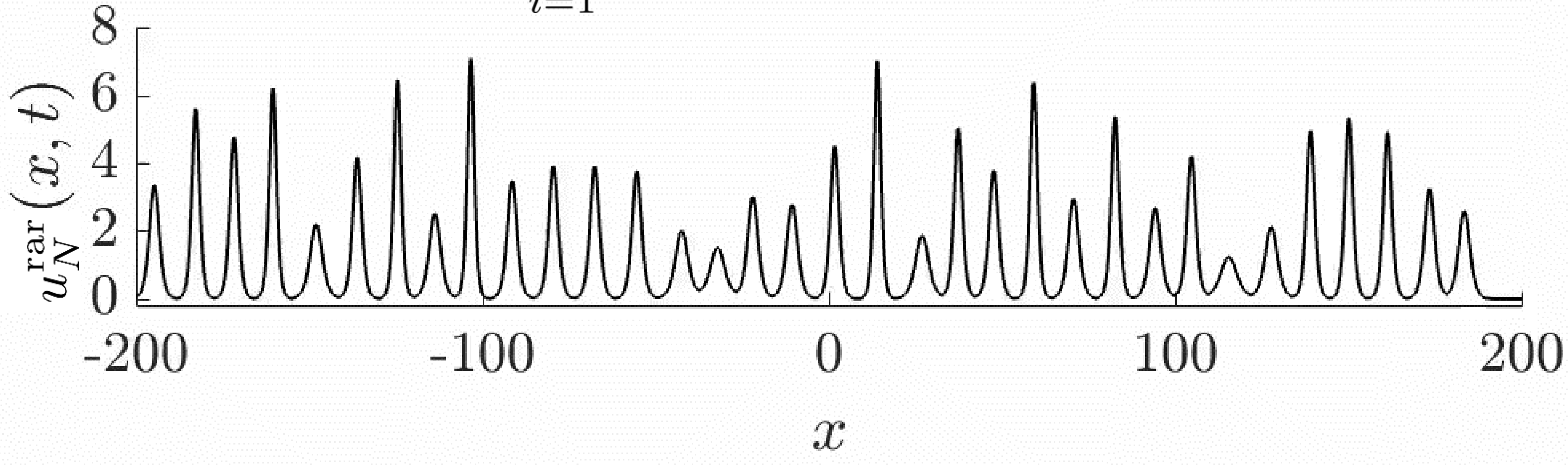

- Spectral support: $\Gamma = \bigcup_{i=0}^{N} [\gamma_{2i}, \gamma_{2i+1}] \subset \mathbb{R}^+ \ .$

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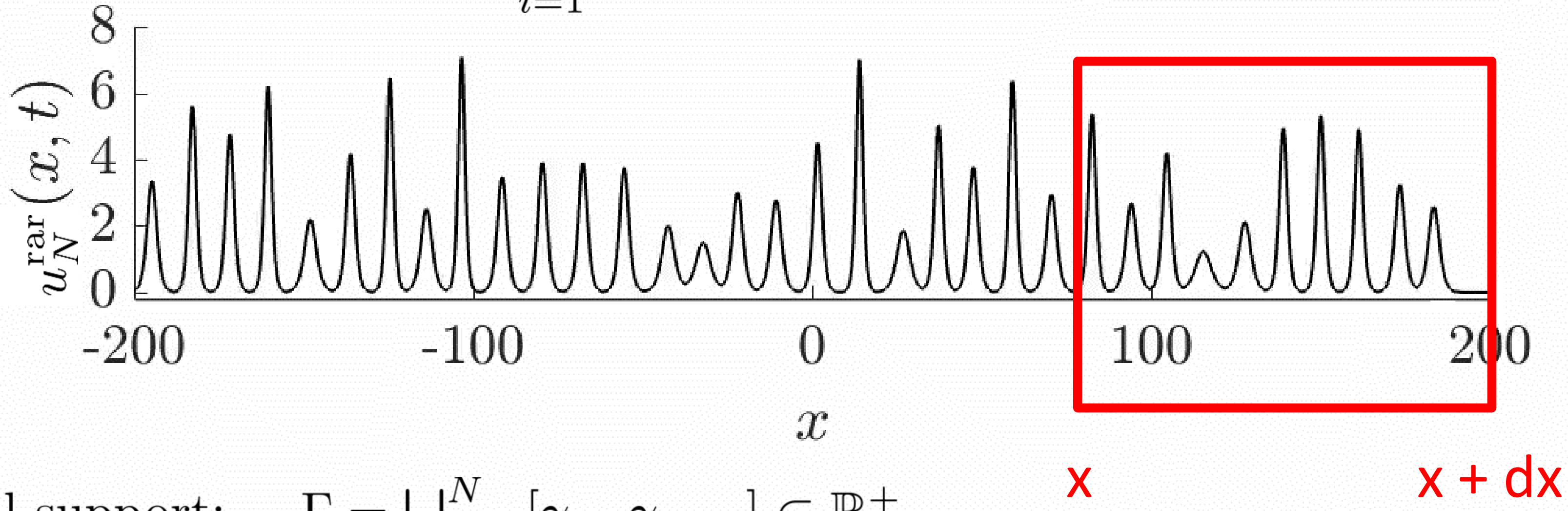

- Spectral support: $\Gamma = \bigcup_{i=0}^{N} [\gamma_{2i}, \gamma_{2i+1}] \subset \mathbb{R}^+$.

- Spectral density of states (DOS): $\rho^{\mathrm{rar}} : \Gamma \times \mathbb{R}^2 \to \mathbb{R}^+$

$$\rho^{\mathrm{rar}}(\eta; x, t)\mathrm{d}\eta\mathrm{d}x \;=\; \#\text{ of solitons at } t \text{ in } [\eta, \eta + \mathrm{d}\eta] \times [x, x + \mathrm{d}x]$$

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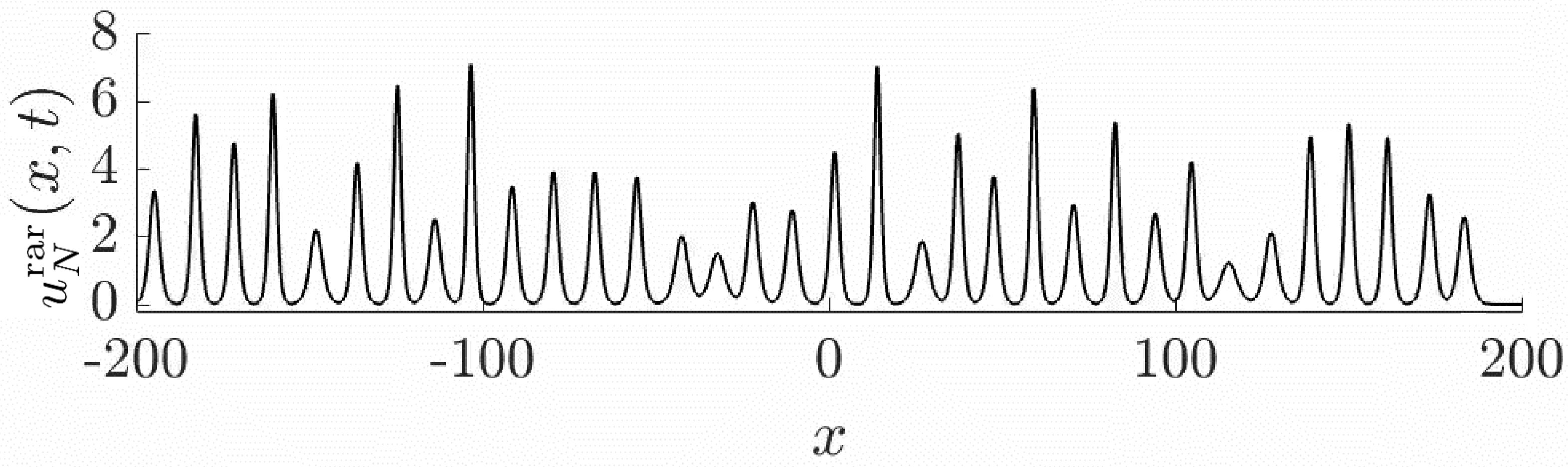

- Condition for the gas to be rarefied

$$\alpha(x,t) = \int_\Gamma \mathrm{d}\eta \; \rho^{\mathrm{rar}}(\eta; x, t) \ll \gamma_0 \; .$$

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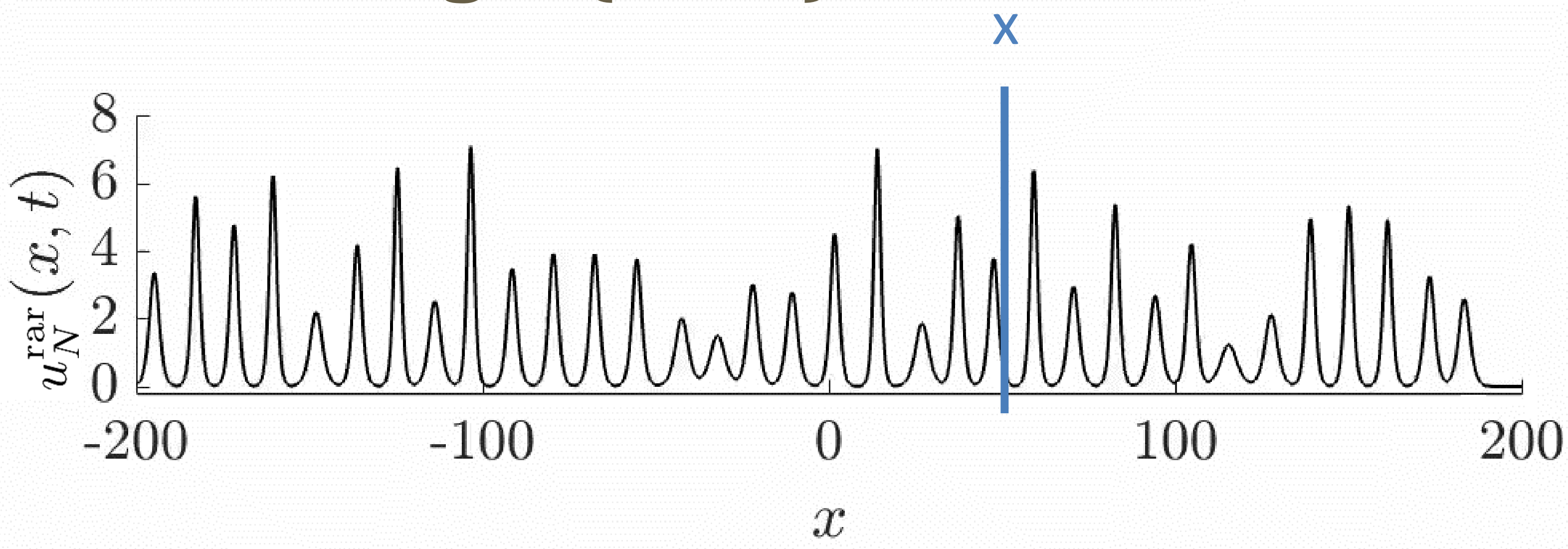

- Condition for the gas to be rarefied

$$\alpha(x,t) = \int_\Gamma \mathrm{d}\eta \; \rho^{\rm rar}(\eta;x,t) \ll \gamma_0 \; .$$

- Spectral flux density: $f^{\rm rar} : \Gamma \times \mathbb{R}^2 \to \mathbb{R}$

$$f^{\rm rar}(\eta;x,t)\mathrm{d}\eta\mathrm{d}t \;=\; \#\text{ of solitons crossing } x \text{ in } [\eta,\eta+\mathrm{d}\eta]\times[t,t+\mathrm{d}t] \; .$$

Zakharov's kinetic equation for solitons

- Isospectrality imposes DOS is only transported over large scales

$$\partial_t \rho^{\mathrm{rar}}(\eta; x, t) + \partial_x f^{\mathrm{rar}}(\eta; x, t) = 0 \ .$$

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$$4\eta^2 \rho^{\mathrm{rar}}(\eta; x, t) \ ?$$

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$$v^{\text{rar}}(\eta; x, t)\rho^{\text{rar}}(\eta; x, t)$$

- Solitons move with effective velocity

$$v^{\text{rar}}(\eta; x, t) \approx 4\eta^2 + \frac{1}{\eta}\int_\Gamma \log\left|\frac{\eta+\mu}{\eta-\mu}\right| \rho^{\text{rar}}(\eta; x, t)\left[4\eta^2 - 4\mu^2\right] \mathrm{d}\mu \ .$$

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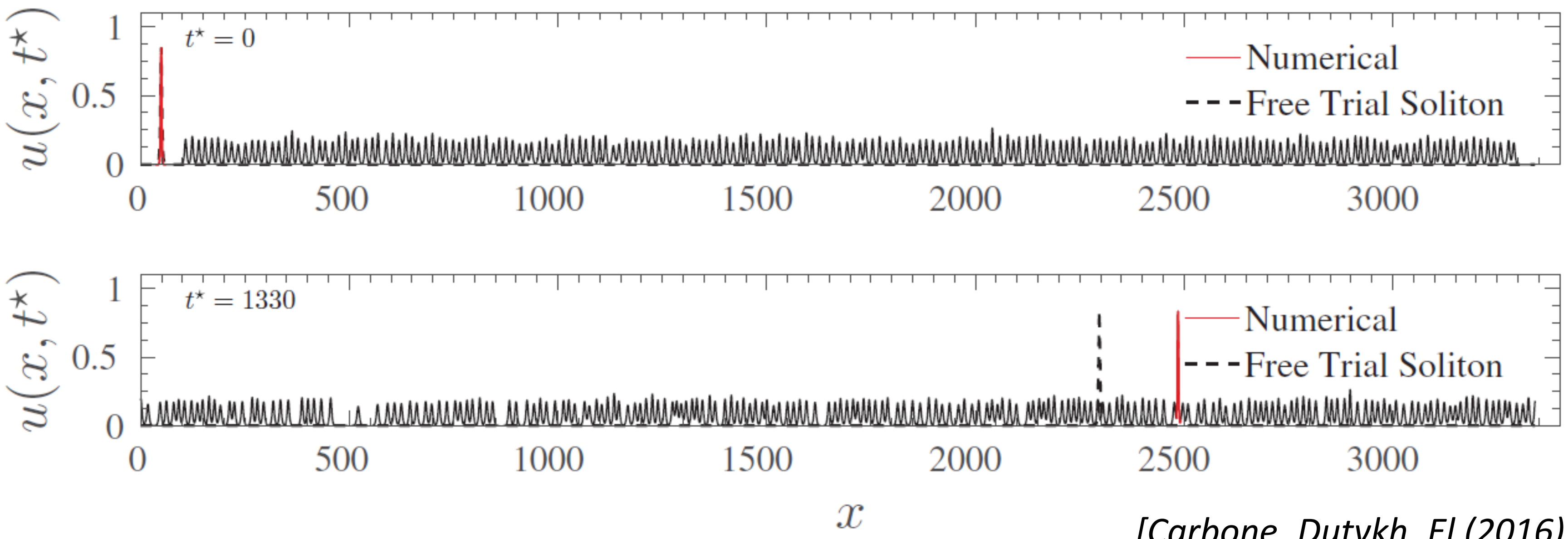
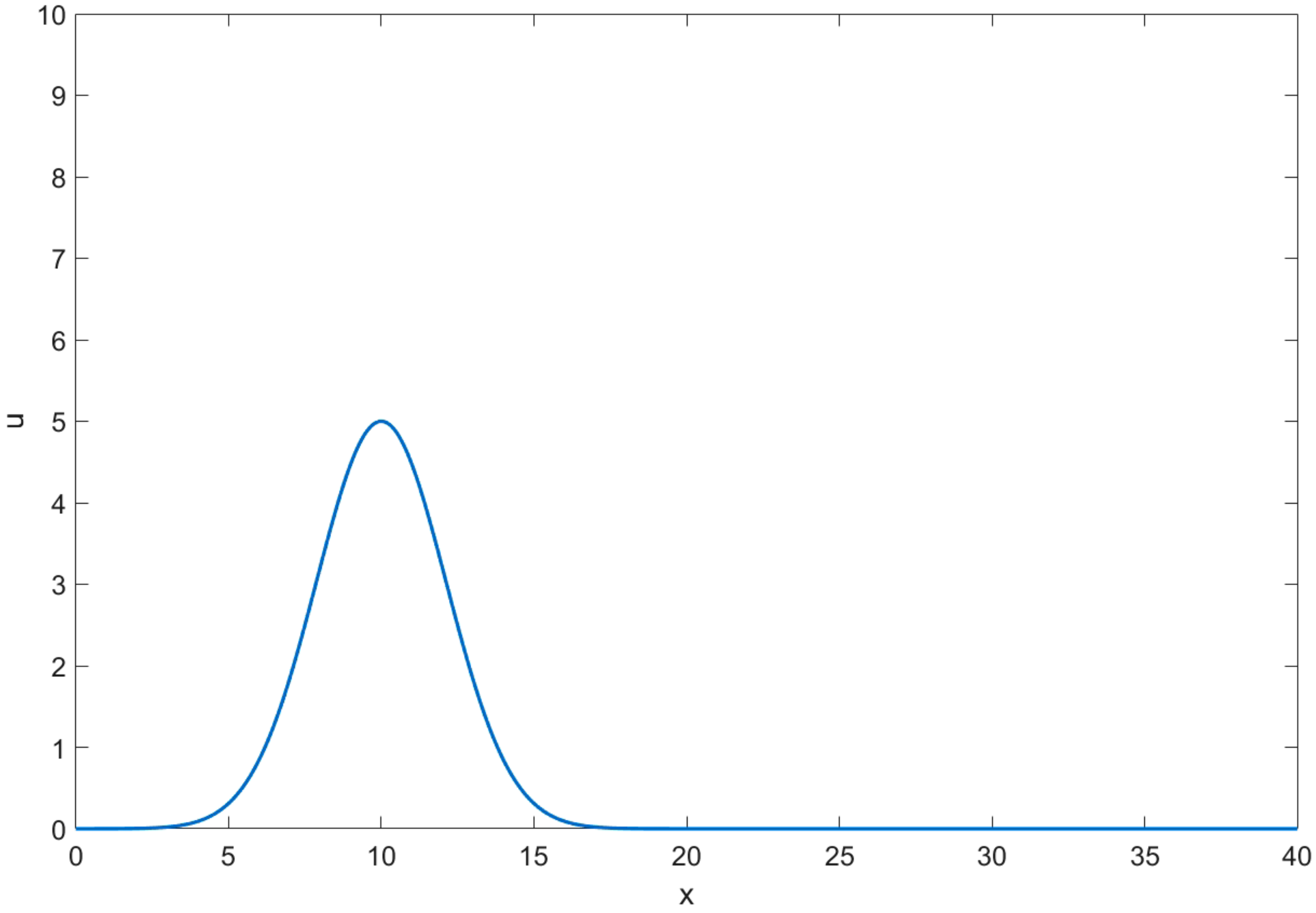


[Carbone, Dutykh, El (2016)]

DOS in dense soliton gases

- Meaning of DOS not as clear in a dense gas.



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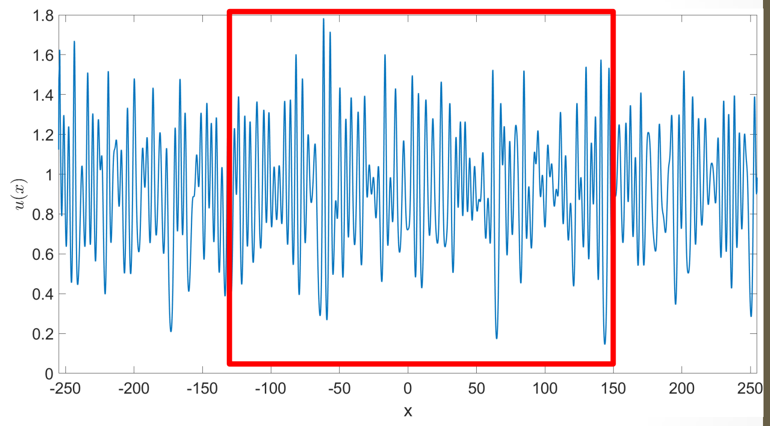
$$u_L(x,t) \approx \sum_{i=1}^{N} 2\eta_i^2 \mathrm{sech}^2\left[\eta_i\left(x - 4\eta_i^2 t - x_i^{\pm}\right)\right] \quad \text{as } t \to \pm\infty.$$

Action coordinate

Angle coordinate



N solitons in L

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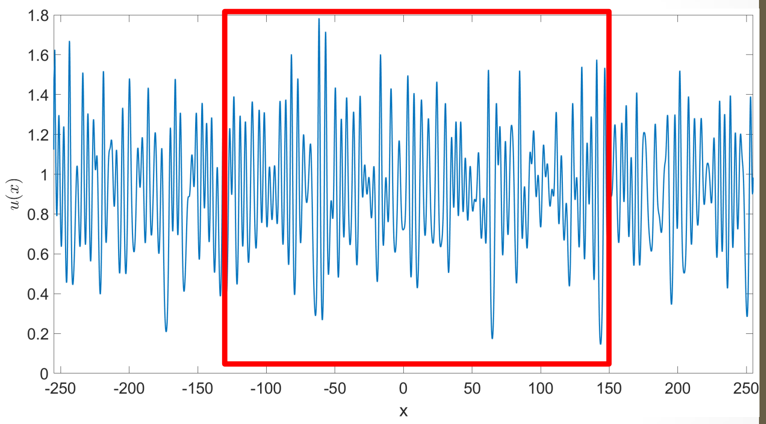

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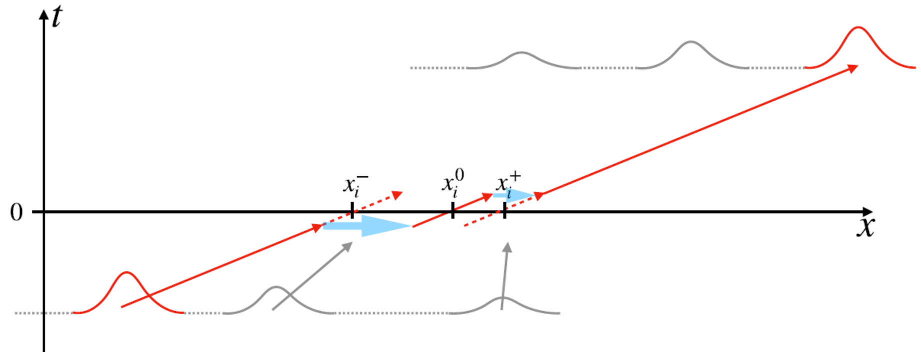
- Relation between asymptotic states given by scattering shift



$$x_i^+ - x_i^- = \sum_j \frac{\mathrm{sgn}(\eta_i - \eta_j)}{\eta_i} \ln\left|\frac{\eta_i + \eta_j}{\eta_i - \eta_j}\right| .$$

Thermodynamics

- $N-$soliton partition function can be formally written as

$$\mathcal{Z}_L = \int \mathcal{D}[u_N] \exp\left(\underbrace{S[u_N]}_{\text{Entropy}} - \underbrace{W[u_N]}_{\text{Generalised Gibbs weight}}\right) .$$

Entropy

Generalised Gibbs weight

$$W = \sum_k \beta_k Q_k$$

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- $N-$soliton in asymptotic coordinates

$$\mathcal{Z}_L = \sum_{N=0}^{\infty} \frac{1}{N!} \int_{\Gamma^N \times \mathbb{R}^N} \prod_{i=1}^{N} \frac{\mathrm{d}p(\eta_i)}{2\pi} \mathrm{d}x_i^- \exp\left[-\sum_{i=1}^{N} w(\eta_i)\right] \chi\left(u_N(x, t=0) < \epsilon_x, x \notin [0, L]\right) .$$

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Soliton bare velocity

$$p(\eta) = 4\eta^2$$

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e.g. $w(\eta) = \sum_k \beta_k h_k(\eta)$

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Constraint / Entropy

Asymptotic space

- Assume solitons are point particles of velocity $p(\eta_i)$ and position $X_i^t = x_i^t + p(\eta_i)t$.

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- Assume: $\forall\ i = 1 \cdots N,\ \ X_i^0 = x_i^0 \in [0, L]$.
- Let i be the leftmost soliton $(x_i^0 = 0)$

$$0 \quad = \quad x_i^{\mathrm{left}} - \frac{1}{\eta_i} \sum_{\eta_j > \eta_i} \log \left| \frac{\eta_i + \eta_j}{\eta_i - \eta_j} \right| \ .$$

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Asymptotic position x_i^-

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Position at t=0

Shifts from faster solitons

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- Let i be the rightmost soliton $(x_i^0 = L)$

$$L \ = \ x_i^{\text{right}} + \frac{1}{\eta_i} \sum_{\eta_j < \eta_i} \log \left| \frac{\eta_i + \eta_j}{\eta_i - \eta_j} \right| \ .$$

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$$0 = x_i^{\text{left}} - \frac{1}{\eta_i} \sum_{\eta_j > \eta_i} \log \left| \frac{\eta_i + \eta_j}{\eta_i - \eta_j} \right| .$$

- Let i be the rightmost soliton $(x_i^0 = L)$

$$L = x_i^{\text{right}} + \frac{1}{\eta_i} \sum_{\eta_j < \eta_i} \log \left| \frac{\eta_i + \eta_j}{\eta_i - \eta_j} \right| .$$

$\Rightarrow$

Asymptotic space is shorter than real space

$$L_i \equiv x_i^{\text{right}} - x_i^{\text{left}}$$
$$= L - \frac{1}{\eta_i} \sum_{j \neq i} \log \left| \frac{\eta_i + \eta_j}{\eta_i - \eta_j} \right| .$$

Asymptotic space density

- Let $L_N(\eta)$ interpolate L_i

$$\mathcal{K}_N(\eta) \equiv \frac{L_N(\eta)}{L} = 1 - \frac{1}{L\eta} \sum_{j=1}^{N} \log \left| \frac{\eta + \eta_j}{\eta - \eta_j} \right| \; .$$

Asymptotic space density

- Let $L_N(\eta)$ interpolate L_i

$$\mathcal{K}_N(\eta) \equiv \frac{L_N(\eta)}{L} = 1 - \frac{1}{L\eta}\sum_{j=1}^{N} \log\left|\frac{\eta+\eta_j}{\eta-\eta_j}\right| \ .$$

- Limit $N \to \infty$, $L \to \infty$, $N/L = \varkappa$

$$\mathcal{K}(\eta) = 1 - \frac{1}{\eta}\int_\Gamma \mathrm{d}\mu \ \rho(\mu) \log\left|\frac{\eta+\mu}{\eta-\mu}\right| \ .$$

Aymptotic space density

DOS

Asymptotic space density

- Let $L_N(\eta)$ interpolate L_i

$$\mathcal{K}_N(\eta) \equiv \frac{L_N(\eta)}{L} = 1 - \frac{1}{L\eta}\sum_{j=1}^{N} \log\left|\frac{\eta+\eta_j}{\eta-\eta_j}\right| .$$

- Limit $N \to \infty$, $L \to \infty$, $N/L = \varkappa$

$$\mathcal{K}(\eta) = 1 - \frac{1}{\eta}\int_\Gamma \mathrm{d}\mu \; \rho(\mu) \log\left|\frac{\eta+\mu}{\eta-\mu}\right| .$$

Aymptotic space density

DOS

$$\frac{\varkappa}{N}\sum_{i=1}^{N}\delta(\eta-\eta_i)$$

$$\langle q_n \rangle = \int_\Gamma \mathrm{d}\eta \; \rho(\eta) h_n(\eta)$$

$h_n(\eta) = Q_n$ for a single soliton η

Asymptotic space density

- Let $L_N(\eta)$ interpolate L_i

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$$\mathcal{K}(\eta) = 1 - \frac{1}{\eta}\int_\Gamma \mathrm{d}\mu\ \rho(\mu) \log\left|\frac{\eta+\mu}{\eta-\mu}\right| .$$

Aymptotic space density

DOS

$$\mathrm{d}x^-(\eta) = \mathcal{K}(\eta)\mathrm{d}x$$

change of metric due to interactions

$$\frac{\varkappa}{N}\sum_{i=1}^{N}\delta(\eta-\eta_i)$$

$$\langle q_n\rangle = \int_\Gamma \mathrm{d}\eta\ \rho(\eta)h_n(\eta)$$

$h_n(\eta) = Q_n$ for a single soliton η

Asymptotic constraint

- $N-$soliton partition function in asymptotic coordinates

$$\mathcal{Z}_L = \sum_{N=0}^{\infty} \frac{1}{N!} \int_{\Gamma^N \times \mathbb{R}^N} \prod_{i=1}^{N} \frac{\mathrm{d}p(\eta_i)}{2\pi} \mathrm{d}x_i^- \exp\left[-\sum_{i=1}^{N} w(\eta_i)\right] \chi\left(u_N(x, t=0) < \epsilon_x, x \notin [0, L]\right) \ .$$

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- Asymptotic constraint

$$\int_{\mathbb{R}^N} \prod_{i=1}^{N} \mathrm{d}x_i^- \chi\left(u_N(x,t=0), x \notin [0,L]\right) \approx \prod_{i=1}^{N} \left(\int_{x_i^{\text{left}}(\eta_i)}^{x_i^{\text{right}}(\eta_i)} \mathrm{d}x^- \right) = L^N \prod_{i=1}^{N} \mathcal{K}_N(\eta_i) \,.$$

Asymptotic constraint

- $N-$soliton partition function in asymptotic coordinates

$$\mathcal{Z}_L = \sum_{N=0}^{\infty} \frac{1}{N!} \int_{\Gamma^N \times \mathbb{R}^N} \prod_{i=1}^{N} \frac{\mathrm{d}p(\eta_i)}{2\pi} \mathrm{d}x_i^- \exp\left[- \sum_{i=1}^{N} w(\eta_i) \right] \chi\left(u_N(x,t=0) < \epsilon_x, x \notin [0,L]\right) \ .$$

- Asymptotic constraint

$$\int_{\mathbb{R}^N} \prod_{i=1}^{N} \mathrm{d}x_i^- \chi\left(u_N(x,t=0), x \notin [0,L]\right) \approx \prod_{i=1}^{N} \left(\int_{x_i^{\mathrm{left}}(\eta_i)}^{x_i^{\mathrm{right}}(\eta_i)} \mathrm{d}x^- \right) = L^N \prod_{i=1}^{N} \mathcal{K}_N(\eta_i) \ .$$

- Putting everything in the exponential

$$\mathcal{Z}_L = \sum_{N=0}^{\infty} \int_{\Gamma^N} \prod_{i=1}^{N} \mathrm{d}\eta_i \exp\left\{ - \sum_{i=0}^{N} \left[w(\eta_i) - \log\left(\frac{4\eta_i}{\pi}\right) - \log\left[\mathcal{K}_N(\eta_i)\right] - 1 + \log \varkappa \right] \right\} \ .$$

Asymptotic constraint

- $N-$soliton partition function in asymptotic coordinates

$$\mathcal{Z}_L = \sum_{N=0}^{\infty} \frac{1}{N!} \int_{\Gamma^N \times \mathbb{R}^N} \prod_{i=1}^{N} \frac{\mathrm{d}p(\eta_i)}{2\pi} \mathrm{d}x_i^- \exp\left[-\sum_{i=1}^{N} w(\eta_i)\right] \chi\left(u_N(x,t=0) < \epsilon_x, x \notin [0,L]\right) .$$

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$$\int_{\mathbb{R}^N} \prod_{i=1}^{N} \mathrm{d}x_i^- \chi\left(u_N(x,t=0), x \notin [0,L]\right) \approx \prod_{i=1}^{N} \left(\int_{x_i^{\mathrm{left}}(\eta_i)}^{x_i^{\mathrm{right}}(\eta_i)} \mathrm{d}x^- \right) = L^N \prod_{i=1}^{N} \mathcal{K}_N(\eta_i) .$$

- Putting everything in the exponential

Jacobian

Prefactor

$$\mathcal{Z}_L = \sum_{N=0}^{\infty} \int_{\Gamma^N} \prod_{i=1}^{N} \mathrm{d}\eta_i \exp\left\{-\sum_{i=0}^{N}\left[w(\eta_i) - \log\left(\frac{4\eta_i}{\pi}\right) - \log\left[\mathcal{K}_N(\eta_i)\right] - 1 + \log \varkappa \right]\right\} .$$

Gibbs weight

Constraint

Asymptotic constraint

- $N-$soliton partition function in asymptotic coordinates

$$\mathcal{Z}_L = \sum_{N=0}^{\infty} \frac{1}{N!} \int_{\Gamma^N \times \mathbb{R}^N} \prod_{i=1}^{N} \frac{\mathrm{d}p(\eta_i)}{2\pi} \mathrm{d}x_i^- \exp\left[-\sum_{i=1}^{N} w(\eta_i)\right] \chi\left(u_N(x,t=0) < \epsilon_x, x \notin [0,L]\right) \,.$$

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$$\int_{\mathbb{R}^N} \prod_{i=1}^{N} \mathrm{d}x_i^- \chi\left(u_N(x,t=0), x \notin [0,L]\right) \approx \prod_{i=1}^{N} \left(\int_{x_i^{\mathrm{left}}(\eta_i)}^{x_i^{\mathrm{right}}(\eta_i)} \mathrm{d}x^- \right) = L^N \prod_{i=1}^{N} \mathcal{K}_N(\eta_i)\,.$$

- Putting everything in the exponential

$$\mathcal{Z}_L = \sum_{N=0}^{\infty} \int_{\Gamma^N} \prod_{i=1}^{N} \mathrm{d}\eta_i \exp\left\{-L \int_{\Gamma} \mathrm{d}\eta \, \tilde{\rho}(\eta) \left[w(\eta) - \log\left[\frac{4\eta \mathcal{K}_N(\eta)}{\pi}\right] - 1 + \log \varkappa \right]\right\}.$$

Empirical DOS

Thermodynamic equilibrium

- Thermodynamic limit: large deviations theory *[Varadhan (1966), Touchette (2009)]*

$$\mathcal{Z}_L \asymp \exp\left(-L\mathcal{F}^{\mathrm{MF}}[\rho^*(\eta)]\right) \;,$$

with

$$\mathcal{F}^{\mathrm{MF}}[\rho(\eta)] = \int_\Gamma \mathrm{d}\eta\; \rho(\eta)\left[w(\eta) - \log\left[\frac{4\eta\mathcal{K}(\eta)}{\pi}\right] - 1 + \log\rho(\eta)\right] \;.$$

Thermodynamic equilibrium

- Thermodynamic limit: large deviations theory

[Varadhan (1966), Touchette (2009)]

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Configuration entropy

[Sanov (1961)]

Thermodynamic equilibrium

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- Minimisation condition (Yang-Yang equation)

$$0 = \left.\frac{\delta\mathcal{F}^{\mathrm{MF}}[\rho]}{\delta\rho(\eta)}\right|_{\rho=\rho^*} \quad \Rightarrow \quad \log\frac{4\eta\mathcal{K}(\eta)}{\pi\rho(\eta)} = w(\eta) + \int_\Gamma \mathrm{d}\mu \ \log\left|\frac{\eta+\mu}{\eta-\mu}\right| \frac{\rho(\mu)}{\mu\mathcal{K}(\mu)} \ .$$

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- Free energy (spatial) density

$$\mathcal{F} \equiv \mathcal{F}^{\mathrm{MF}}[\rho(\eta)] = -\int_\Gamma \mathrm{d}\mu\ \frac{\rho(\mu)}{\mathcal{K}(\mu)} \ .$$

Thermodynamic equilibrium

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$$\mathcal{F} \equiv \mathcal{F}^{\mathrm{MF}}[\rho(\eta)] = -\int_\Gamma \mathrm{d}\mu\ \frac{\rho(\mu)}{\mathcal{K}(\mu)} = \int_\Gamma \frac{\mathrm{d}p(\mu)}{2\pi}F(\mu) \ .$$

Thermodynamic equilibrium (alternative notations)

- Minimisation condition (Yang-Yang equation)

$$\epsilon(\eta) = w(\eta) - \int_\Gamma \frac{\mathrm{d}p(\mu)}{2\pi\mu} \log\left|\frac{\eta+\mu}{\eta-\mu}\right| F(\mu)\ .$$

"pseudo-energy"

$$\epsilon = \log \frac{4\eta\mathcal{K}(\eta)}{\pi\rho(\eta)}$$

free energy density

$$F = -e^{-\epsilon(\eta)}$$

Thermodynamic equilibrium (alternative notations)

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"pseudo-energy"

free energy density

$$\epsilon = \log \frac{4\eta\mathcal{K}(\eta)}{\pi\rho(\eta)}$$

$$F = -e^{-\epsilon(\eta)}$$

- Occupation function

$$n(\eta) = \left.\frac{\mathrm{d}F}{\mathrm{d}\epsilon}\right|_{\epsilon=\epsilon(\eta)} = e^{-\epsilon(\eta)}\,.$$

Maxwell-Boltzmann

Thermodynamic equilibrium (alternative notations)

- Minimisation condition (Yang-Yang equation)

$$\epsilon(\eta) = w(\eta) - \int_\Gamma \frac{\mathrm{d}p(\mu)}{2\pi\mu} \log\left|\frac{\eta+\mu}{\eta-\mu}\right| F(\mu)\ .$$

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$$F = -e^{-\epsilon(\eta)}$$

- Occupation function

$$n(\eta) = \left.\frac{\mathrm{d}F}{\mathrm{d}\epsilon}\right|_{\epsilon=\epsilon(\eta)} = e^{-\epsilon(\eta)} = \frac{\pi\rho(\eta)}{4\eta\mathcal{K}(\eta)}$$

Density of solitons in the asymptotic space

Maxwell-Boltzmann

Thermodynamic quantities

- Entropy density of the soliton gas

$$\mathcal{S} = \mathcal{W} - \mathcal{F}$$

$$\int_\Gamma \mathrm{d}\eta \; \rho(\eta) w(\eta)$$

Thermodynamic quantities

- Entropy density of the soliton gas

$$\mathcal{S} = \mathcal{W} - \mathcal{F} = \int_\Gamma \mathrm{d}\eta \; \rho(\eta) \left[1 - \log n(\eta)\right] \; .$$

Thermodynamic quantities

- Entropy density of the soliton gas

$$\mathcal{S} = \mathcal{W} - \mathcal{F} = \int_\Gamma \mathrm{d}\eta \; \rho(\eta) \left[1 - \log n(\eta)\right] \; .$$

- Thermodynic averages

$$\langle q_n \rangle = \frac{\partial \mathcal{F}}{\partial \beta_n} \; .$$

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$$\langle q_n \rangle = \frac{\partial \mathcal{F}}{\partial \beta_n} = \int_\Gamma \frac{\mathrm{d}p(\eta)}{2\pi} \; n(\eta) \partial_{\beta_n} \epsilon(\eta) \; ,$$

$$\mathcal{F} = -\int_\Gamma \frac{\mathrm{d}p(\eta)}{2\pi} F(\eta) \; , \qquad F = -e^{-\epsilon(\eta)} \; , \qquad n(\eta) = \left. \frac{\mathrm{d}F}{\mathrm{d}\epsilon} \right|_{\epsilon = \epsilon(\eta)} \; .$$

Aside: dressing operation

- Differentiating the Yang-Yang equation

$$\epsilon(\eta) = w(\eta) - \int_\Gamma \frac{\mathrm{d}p(\mu)}{2\pi\mu} \log\left|\frac{\eta+\mu}{\eta-\mu}\right| F(\mu) \ ,$$

$$w(\eta) = \sum_n \beta_n h_n(\eta)$$

$$\partial_{\beta_n}\epsilon(\eta) = h_n(\eta) - \int_\Gamma \frac{\mathrm{d}p(\mu)}{2\pi\mu} \log\left|\frac{\eta+\mu}{\eta-\mu}\right| n(\mu)\partial_{\beta_n}\epsilon(\mu) \ .$$

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- Differentiating the Yang-Yang equation

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$$\partial_{\beta_n}\epsilon(\eta) = h_n(\eta) - \int_\Gamma \frac{\mathrm{d}p(\mu)}{2\pi\mu} \log\left|\frac{\eta+\mu}{\eta-\mu}\right| n(\mu)\partial_{\beta_n}\epsilon(\mu) \ .$$

- Let $f : \Gamma \to \mathbb{R}$, we define the dressed function $f^{\mathrm{dr}} : \Gamma \to \mathbb{R}$ from this Fredholm equation of the 2nd kind

$$f^{\mathrm{dr}}(\eta) = f(\eta) + \int_\Gamma \frac{\mathrm{d}\mu}{2\pi} \varphi(\eta;\mu) n(\mu) f^{\mathrm{dr}}(\mu) \ .$$

$$\partial_{\beta_n}\epsilon(\eta) = h_n^{\mathrm{dr}}(\eta)$$

$$\varphi(\eta;\mu) = 8\log\left|\frac{\eta-\mu}{\eta+\mu}\right|$$

Aside: dressing operation

- Example of dressing from earlier in the lecture

$$\mathcal{K}(\eta) = 1 - \frac{1}{\eta}\int_\Gamma \mathrm{d}\mu\ \rho(\mu)\log\left|\frac{\eta+\mu}{\eta-\mu}\right| ,$$

$$n(\eta) = \frac{\pi\rho(\eta)}{4\eta\mathcal{K}(\eta)}$$

$$\eta\mathcal{K}(\eta) = \eta + \int_\Gamma \frac{\mathrm{d}\mu}{2\pi}\ \varphi(\eta;\mu)n(\mu)\mu\mathcal{K}(\mu) .$$

$$\eta\mathcal{K}(\eta) = \eta^{\mathrm{dr}}(\eta) = h_0^{\mathrm{dr}}(\eta) = \partial_{\beta_0}\epsilon(\eta)$$

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$$\eta\mathcal{K}(\eta) = \eta^{\mathrm{dr}}(\eta) = h_0^{\mathrm{dr}}(\eta) = \partial_{\beta_0}\epsilon(\eta)$$

- Useful property of the dressing

$$\int_\Gamma \mathrm{d}\mu\ g(\mu)n(\mu)f^{\mathrm{dr}}(\mu) = \int_\Gamma \mathrm{d}\mu\ g^{\mathrm{dr}}(\mu)n(\mu)f(\mu) .$$

Thermodynamic quantities

- Thermodynic averages

$$\langle q_n \rangle = \frac{\partial \mathcal{F}}{\partial \beta_n} = \int_\Gamma \frac{\mathrm{d}p(\eta)}{2\pi} \, n(\eta) \partial_{\beta_n} \epsilon(\eta) \ .$$

Thermodynamic quantities

- Thermodynic averages

$$\langle q_n \rangle = \frac{\partial \mathcal{F}}{\partial \beta_n} = \int_\Gamma \frac{\mathrm{d}p(\eta)}{2\pi}\, n(\eta) h_n^{\mathrm{dr}}(\eta)$$

$$\eta \mathcal{K}(\eta) = \eta^{\mathrm{dr}}(\eta)$$

and

$$n(\eta) = \frac{\pi \rho(\eta)}{4\eta \mathcal{K}(\eta)}$$

$$= \int_\Gamma \mathrm{d}\eta\, \frac{4\eta^{\mathrm{dr}}(\eta)}{\pi} n(\eta) h_n(\eta)$$

$$= \int_\Gamma \mathrm{d}\eta\, \rho(\eta) h_n(\eta)\ .$$

Thermodynamic quantities

- Thermodynic averages

$$\langle q_n \rangle = \frac{\partial \mathcal{F}}{\partial \beta_n} = \int_\Gamma \frac{\mathrm{d}p(\eta)}{2\pi}\; n(\eta) h_n^{\mathrm{dr}}(\eta)$$

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$$= \int_\Gamma \mathrm{d}\eta\; \frac{4\eta^{\mathrm{dr}}(\eta)}{\pi} n(\eta) h_n(\eta)$$

$$= \int_\Gamma \mathrm{d}\eta\; \rho(\eta) h_n(\eta)\ .$$

- A few examples:

$$\langle q_0 \rangle = 4 \int_\Gamma \mathrm{d}\eta\; \rho(\eta)\eta = \langle u \rangle \qquad \langle q_1 \rangle = \frac{16}{3} \int_\Gamma \mathrm{d}\eta\; \rho(\eta)\eta^3 = \langle u^2 \rangle$$

$$\langle q_2 \rangle = \frac{32}{5} \int_\Gamma \mathrm{d}\eta\; \rho(\eta)\eta^5 = \left\langle \frac{u_x^2}{2} + u^3 \right\rangle$$

Thermodynamic quantities

- Static covariance matrix

$$\mathsf{C}_{ab} \equiv \int_{\mathbb{R}} \mathrm{d}x \left(\langle q_a(x) q_b(0) \rangle - \langle q_a(x) \rangle \langle q_b(0) \rangle \right) = -\frac{\partial^2 \mathcal{F}}{\partial \beta_a \partial \beta_b} \, .$$

Thermodynamic quantities

- Static covariance matrix

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$$\partial_\beta \int_\Gamma \mathrm{d}\mu \ g(\mu) n(\mu) f^{\mathrm{dr}}(\mu) = \int_\Gamma \mathrm{d}\mu \ g^{\mathrm{dr}}(\mu) \partial_\beta n(\mu) f^{\mathrm{dr}}(\mu)$$

$$\mathsf{C}_{ab} = \int_\Gamma \mathrm{d}\eta \, \rho(\eta) \theta(\eta) h_a^{\mathrm{dr}}(\eta) h_b^{\mathrm{dr}}(\eta) \ .$$

Thermodynamic quantities

- Static covariance matrix

$$\mathsf{C}_{ab} \equiv \int_{\mathbb{R}} \mathrm{d}x \, \left(\langle q_a(x) q_b(0)\rangle - \langle q_a(x)\rangle\langle q_b(0)\rangle\right) = -\frac{\partial^2 \mathcal{F}}{\partial\beta_a \partial\beta_b} \, .$$

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$$\mathsf{C}_{ab} = \int_\Gamma \mathrm{d}\eta \, \rho(\eta)\theta(\eta) h_a^{\mathrm{dr}}(\eta) h_b^{\mathrm{dr}}(\eta) \, .$$

Statistical factor

$$\theta(\eta) = -\frac{\partial_\epsilon n(\eta)}{n(\eta)} = 1$$

Thermodynamic quantities

- Static covariance matrix

$$\mathsf{C}_{ab} \equiv \int_{\mathbb{R}} \mathrm{d}x \left(\langle q_a(x) q_b(0) \rangle - \langle q_a(x) \rangle \langle q_b(0) \rangle \right) = -\frac{\partial^2 \mathcal{F}}{\partial \beta_a \partial \beta_b} \, .$$

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Statistical factor

$$\theta(\eta) = -\frac{\partial_\epsilon n(\eta)}{n(\eta)} = 1$$

$$h^{\mathrm{dr}}(\eta) = h(\eta) + \int_\Gamma \frac{\mathrm{d}\mu}{2\pi} \, \varphi(\eta; \mu) n(\mu) h^{\mathrm{dr}}(\mu)$$

Thermodynamic quantities

- Static covariance matrix

$$\mathsf{C}_{ab} \equiv \int_{\mathbb{R}} \mathrm{d}x \left(\langle q_a(x) q_b(0) \rangle - \langle q_a(x) \rangle \langle q_b(0) \rangle \right) = -\frac{\partial^2 \mathcal{F}}{\partial \beta_a \partial \beta_b}$$

$$= \int_{\Gamma} \mathrm{d}\eta \, \rho(\eta) \theta(\eta) h_a^{\mathrm{dr}}(\eta) h_b^{\mathrm{dr}}(\eta) \; .$$

	MB	FD	BE	Simulations
$\mathsf{C}_{00}^{\mathrm{DC}}$	0.0235	-2.28	2.32	0.022 ± 0.003
$\mathsf{C}_{01}^{\mathrm{DC}}$	0.027	-3.18	3.23	0.024 ± 0.004
$\mathsf{C}_{11}^{\mathrm{DC}}$	0.042	-4.48	4.56	0.039 ± 0.005
$\mathsf{C}_{00}^{\mathrm{U}}$	0.22	0.028	0.41	0.2 ± 0.03
$\mathsf{C}_{01}^{\mathrm{U}}$	0.28	0.072	0.49	0.23 ± 0.04
$\mathsf{C}_{11}^{\mathrm{U}}$	0.39	0.12	0.66	0.36 ± 0.05
$\mathsf{C}_{00}^{\mathrm{L}}$	0.2	-0.05	0.45	0.2 ± 0.01
$\mathsf{C}_{01}^{\mathrm{L}}$	0.25	-0.03	0.54	0.23 ± 0.01
$\mathsf{C}_{11}^{\mathrm{L}}$	0.36	-0.03	0.75	0.34 ± 0.02

Multiphase finite gap solutions of KdV

[Flaschka, Forest, McLaughlin (1980)]

- N-phase solutions associated with band spectrum $\lambda \in [\lambda_0, \lambda_1] \cup \cdots \cup [\lambda_{2N}, +\infty[$

$$u(x,t) = \Lambda + \Phi - 2\log\left[\Theta\left(\boldsymbol{\theta}(x,t);\mathbf{B}\right)\right]_{xx} \quad .$$

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Riemann Theta

Riemann period matrix

$$u(x,t) = \Lambda + \Phi - 2\log\left[\Theta\left(\boldsymbol{\theta}(x,t);\mathbf{B}\right)\right]_{xx} \quad .$$

Constants (depend on endpoints)

Phase vector

Multiphase finite gap solutions of KdV

[Flaschka, Forest, McLaughlin (1980)]

- N-phase solutions associated with band spectrum $\lambda \in [\lambda_0, \lambda_1] \cup \cdots \cup [\lambda_{2N}, +\infty[$

Riemann Theta

Riemann period matrix

$$u(x,t) = \Lambda + \Phi - 2\log\left[\Theta\left(\boldsymbol{\theta}(x,t); \mathbf{B}\right)\right]_{xx} \quad .$$

Constants (depend on endpoints)

Phase vector

- Introduce the two-sheeted hyperelliptic Riemann surface $\mathcal{R}$

$$\mathcal{R}: \quad R^2(z) = \prod_{j=0}^{2N} (z - \lambda_j) \ , \quad z \in \mathbb{C} \ .$$

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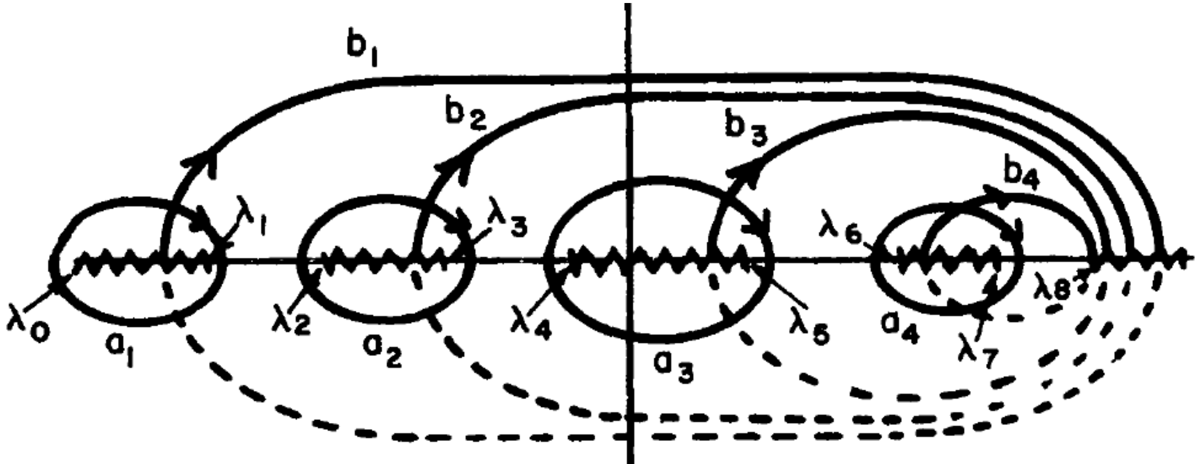
- Canonical homology basis



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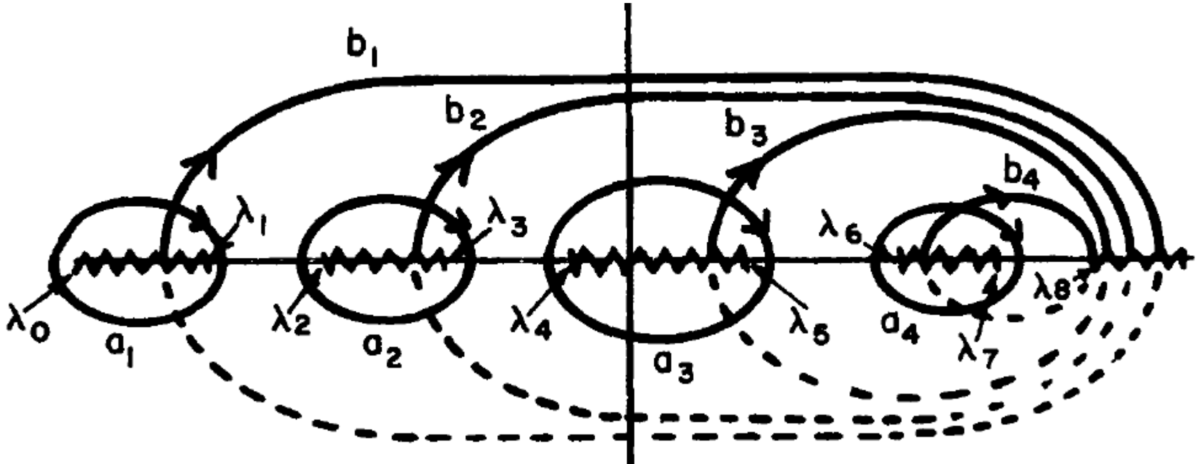

- Basis of meromorphic differentials on $\mathcal{R}$

$$\phi_j = \sum_{k=0}^{N-1} c_{jk} \frac{z^k}{R(z)} \mathrm{d}z \ ,$$

$$\oint_{a_k} \phi_j = \delta_{jk} \ .$$

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- Nonlinear dispersion relations (NDRs)

$$\mathbf{k} = 4\pi i \mathbf{B}^{-1}\mathbf{c}^{(N)} \; , \quad \text{and} \quad \boldsymbol{\omega} = 8\pi i \mathbf{B}^{-1}\left[\Lambda \mathbf{c}^{(N)} + 2\mathbf{c}^{(N-1)}\right] \; ,$$

with $[\mathbf{c}^{(M)}]_j = c_{jM}$.

Multiphase solutions in a nutshell

- Multiphase solution of KdV can be formally written as

$$u(x,t) = F_N(\theta_1, \cdots, \theta_n) \ , \quad \text{with} \quad \theta_j = k_j x + \omega_j t + \theta_j^0 \ .$$

$$F_N(\theta_1, \cdots, \theta_j + 2\pi, \cdots \theta_n) = F_N(\theta_1, \cdots, \theta_j, \cdots \theta_n)$$

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- Nonlinear superposition of waves: analogy with Fourier series

$$u_t + \epsilon u u_x + u_{xxx} = 0$$

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NDRs

$\epsilon \to 0$

$\sum_{j=1}^{N} c_j e^{\left[i\left(k_j x + \omega_j t + \theta_j^0\right)\right]}$

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- Main question: how to construct a soliton gas and what are its NDRs?

Thermodynamic limit of finite gap solutions

[El (2003)]

- Solitonic limit: collapse all bands to points $-\eta_j^2 = \frac{\lambda_{2j}+\lambda_{2j+1}}{2}$

$$\lambda_{2j} \to -\eta_j^2 \ , \quad \text{and} \quad \lambda_{2j+1} \to -\eta_j^2 \ , \quad j = 1, 2 \cdots , N \ ,$$

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$$k_j \to 0 \ , \quad \omega_j \to 0 \ , \quad \text{while} \quad \frac{1}{2\pi} \sum_{j=1}^{N} k_j = \alpha \ , \quad \frac{1}{2\pi} \sum_{j=1}^{N} \omega_j = \beta \ .$$

Density of waves — Wave flux

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- Spectral scaling

$$k_j \sim \omega_j \sim \frac{1}{N} \quad \Rightarrow \quad |\text{gap}_j| \sim \frac{-1}{\log|\text{band}_j|} \sim \frac{1}{N} \ .$$

Thermodynamic nonlinear dispersion relations

[El (2003)]

- Recall the NDRs

$$\mathbf{k} = 4\pi i \mathbf{B}^{-1}\mathbf{c}^{(N)} \ , \quad \text{and} \quad \boldsymbol{\omega} = 8\pi i \mathbf{B}^{-1}\left[\Lambda \mathbf{c}^{(N)} + 2\mathbf{c}^{(N-1)}\right] \ .$$

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Log band width

- Asymptotic behaviour of $\mathbf{c}$ and $\mathbf{B}$

$$B_{jj} \sim \frac{i}{\pi}N\tau(\eta_j) \ , \quad B_{jk} \sim \frac{i}{\pi}\log\left|\frac{\eta_j + \eta_k}{\eta_j - \eta_k}\right| \ ,$$

$$c_{jN} \sim -\frac{\eta_j}{2\pi} \ , \quad \Lambda c_{jN} + 2c_{j(N-1)} \sim \frac{\eta_j^3}{\pi} \ .$$

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Log band width

- Asymptotic behaviour of the NDRs at large N

$$\sum_{j=1}^{N} \frac{1}{N} \log\left|\frac{\eta_j + \eta_k}{\eta_j - \eta_k}\right| \varkappa(\eta_j) + \varkappa(\eta_k)\tau(\eta_k) = 2\pi\eta_k \ ,$$

$$\sum_{j=1}^{N} \frac{1}{N} \log\left|\frac{\eta_j + \eta_k}{\eta_j - \eta_k}\right| \nu(\eta_j) + \nu(\eta_k)\tau(\eta_k) = 8\pi\eta_k^3 \ .$$

Thermodynamic nonlinear dispersion relations

[El (2003)]

- Introduce the DOS $\rho(\eta)$ and the spectral flux density $f(\eta)$

$$\rho : \mathbb{R}^+ \to \mathbb{R}^+ \ , \quad \rho(\eta) = \frac{1}{2\pi}\varkappa(\eta)\xi(\eta) \ , \quad \text{so that} \quad \frac{1}{2\pi}\sum_{j=1}^{M<N} k_j \to \int_{\eta_0}^{\eta} \rho(\mu)\mathrm{d}\mu \ ,$$

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- Thermodynamic NDRs with the spectral scaling function $\sigma(\eta) = \tau(\eta)/\xi(\eta) \; > \; 0$

$$\int_\Gamma \log\left|\frac{\eta+\mu}{\eta-\mu}\right| \rho(\mu)\mathrm{d}\mu \; + \; \sigma(\eta)\rho(\eta) = \eta \; ,$$

$$\int_\Gamma \log\left|\frac{\eta+\mu}{\eta-\mu}\right| f(\mu)\mathrm{d}\mu \; + \; \sigma(\eta)f(\eta) = 4\eta^3 \; .$$

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- Thermodynamic NDRs with the spectral scaling function $\sigma(\eta) = \tau(\eta)/\xi(\eta) > 0$

$$\int_\Gamma \log\left|\frac{\eta+\mu}{\eta-\mu}\right| \rho(\mu)\mathrm{d}\mu + \sigma(\eta)\rho(\eta) = \eta \quad \Rightarrow \quad \eta\mathcal{K}(\eta) = (\eta)^{\mathrm{dr}}(\eta) = \sigma(\eta)\rho(\eta) \,,$$

$$\int_\Gamma \log\left|\frac{\eta+\mu}{\eta-\mu}\right| f(\mu)\mathrm{d}\mu + \sigma(\eta)f(\eta) = 4\eta^3 \quad \Rightarrow \quad \left(4\eta^3\right)^{\mathrm{dr}}(\eta) = \sigma(\eta)f(\eta) \,.$$

⇒ $\sigma(\eta) = \frac{\pi}{4n(\eta)}$

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- Generic form of the effective velocity in GHD

$$v^{\mathrm{eff}}(\eta) = v^{\mathrm{gr}}(\eta) + \int_\Gamma \mathrm{d}\mu \; \varphi(\eta;\mu)\rho(\mu)\left[v^{\mathrm{eff}}(\eta) - v^{\mathrm{eff}}(\mu)\right] \; ,$$

$$v^{\mathrm{gr}}(\eta) = \frac{E'(\eta)}{p'(\eta)} \; , \qquad\qquad v^{\mathrm{eff}}(\eta) = \frac{(E')^{\mathrm{dr}}(\eta)}{(p')^{\mathrm{dr}}(\eta)} \; .$$

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- Integrability: infinite number of conservation laws

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⇨

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From thermodynamics to hydrodynamics

- Conservation of waves

$$\partial_t \mathbf{k} + \partial_x \boldsymbol{\omega} = 0 \ .$$

- Slow modulations of finite gap solutions

$$\mathbf{k} = \mathbf{K}[\boldsymbol{\lambda}(x,t)] \ , \quad \boldsymbol{\omega} = \boldsymbol{\Omega}[\boldsymbol{\lambda}(x,t)] \ .$$

- Thermodynamic spectral limit and leading order in multi-scale expansion

$$\partial_t \rho(\eta; x,t) + \partial_x \left[v^{\mathrm{eff}}(\eta; x,t) \rho(\eta; x,t) \right] = 0 \ ,$$

$$v^{\mathrm{eff}}(\eta; x,t) = 4\eta^2 + \frac{1}{\eta} \int_\Gamma \log \left| \frac{\eta+\mu}{\eta-\mu} \right| \rho(\eta; x,t) [v^{\mathrm{eff}}(\eta; x,t) - v^{\mathrm{eff}}(\eta; x,t)] \mathrm{d}\mu \ .$$

Alternative derivation of GHD equations

[Based on: Doyon, Spohn, Yoshimura (2017)]

- Asymptotic dynamics

$$x_j^-(t) = x_j^-(0) + 4\eta_j^2 t \ ,$$

$$\Rightarrow \quad \partial_t \rho^-(\eta; x^-, t) + 4\eta^2 \partial_{x^-} \rho^-(\eta; x^-, t) = 0 \ .$$

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[Based on: Doyon, Spohn, Yoshimura (2017)]

- Asymptotic dynamics

$$x_j^-(t) = x_j^-(0) + 4\eta_j^2 t \ ,$$

$$\Rightarrow \quad \partial_t \rho^-(\eta; x^-, t) + 4\eta^2 \partial_{x^-} \rho^-(\eta; x^-, t) = 0 \ .$$

- Change of metric: $\mathrm{d}x^-(\eta; x, t) = \mathcal{K}(\eta; x, t)\mathrm{d}x$

$$\partial_t n(\eta; x, t) + v^{\mathrm{eff}}(\eta; x, t)\partial_x n(\eta; x, t) = 0 \ .$$

$$v^{\mathrm{eff}}(\eta; x, t) = 4\eta^2 + \frac{1}{\eta}\int_\Gamma \log\left|\frac{\eta+\mu}{\eta-\mu}\right| \rho(\eta; x, t)[v^{\mathrm{eff}}(\eta; x, t) - v^{\mathrm{eff}}(\eta; x, t)]\mathrm{d}\mu \ .$$

Alternative derivation of GHD equations

[Based on: Doyon, Spohn, Yoshimura (2017)]

- Asymptotic dynamics

$$x_j^-(t) = x_j^-(0) + 4\eta_j^2 t \ ,$$

$$\Rightarrow \quad \partial_t \rho^-(\eta; x^-, t) + 4\eta^2 \partial_{x^-} \rho^-(\eta; x^-, t) = 0 \ .$$

- Change of metric: $\mathrm{d}x^-(\eta; x, t) = \mathcal{K}(\eta; x, t)\mathrm{d}x$

$$\partial_t n(\eta; x, t) + v^{\text{eff}}(\eta; x, t)\partial_x n(\eta; x, t) = 0 \ .$$

$$v^{\text{eff}}(\eta; x, t) = 4\eta^2 + \frac{1}{\eta}\int_\Gamma \log\left|\frac{\eta+\mu}{\eta-\mu}\right| \rho(\eta; x, t)[v^{\text{eff}}(\eta; x, t) - v^{\text{eff}}(\eta; x, t)]\mathrm{d}\mu \ .$$

n (equivalently σ) plays the role of a continuum of Riemann invariants!

GHD as an integrable system of hydrodynamic type

- System of hydrodynamic type in Riemann form

$$\partial_t \lambda_j + v_j(\lambda_0, \cdots, \lambda_n)\partial_x \lambda_j = 0 \quad \longrightarrow \quad \partial_t n(\eta; x, t) + v^{\text{eff}}(\eta; x, t)\partial_x \left[n(\eta; x, t)\right] = 0\,.$$

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- Linear degeneracy

$$\partial_{\lambda_j} v_j = 0 \quad \longrightarrow \quad \frac{\delta v^{\text{eff}}(\eta)}{\delta n(\eta)} = 0 \, , \quad \forall \eta \in \Gamma \, .$$

No shocks in GHD!

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- System of hydrodynamic type in Riemann form

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No shocks in GHD!

- Semi-Hamiltonian property

$$\partial_{\lambda_j} \frac{\partial_{\lambda_k} v_i}{v_k - v_i} = \partial_{\lambda_k} \frac{\partial_{\lambda_j} v_i}{v_j - v_i} \, , \quad i \neq j \neq k$$

$$\int_\Gamma \mathrm{d}\nu \left[\frac{\delta}{\delta n(\nu)} \left(\frac{\delta v^{\text{eff}}(\eta)/\delta n(\mu)}{v^{\text{eff}}(\mu) - v^{\text{eff}}(\eta)}\right)\right] = \int_\Gamma \mathrm{d}\mu \left[\frac{\delta}{\delta n(\mu)} \left(\frac{\delta v^{\text{eff}}(\eta)/\delta n(\nu)}{v^{\text{eff}}(\nu) - v^{\text{eff}}(\eta)}\right)\right] \, .$$

GHD equations are integrable!

GHD as an integrable system of hydrodynamic type

- System in Riemann form

$$\partial_t n(\eta;x,t)+v^{\text{eff}}(\eta;x,t)\partial_x\left[n(\eta;x,t)\right]=0\,.$$

- Linear degeneracy

$$\frac{\delta v^{\text{eff}}(\eta)}{\delta n(\eta)}=0\,,\qquad \forall\eta\in\Gamma\,.$$

No shocks in GHD!

- Semi-Hamiltonian property

$$\int_\Gamma \mathrm{d}\nu\left[\frac{\delta}{\delta n(\nu)}\left(\frac{\delta v^{\text{eff}}(\eta)/\delta n(\mu)}{v^{\text{eff}}(\mu)-v^{\text{eff}}(\eta)}\right)\right]=\int_\Gamma \mathrm{d}\mu\left[\frac{\delta}{\delta n(\mu)}\left(\frac{\delta v^{\text{eff}}(\eta)/\delta n(\nu)}{v^{\text{eff}}(\nu)-v^{\text{eff}}(\eta)}\right)\right]\,.$$

GHD equations are integrable!

- Generalised hodograph transform

$$x-4\eta^2t=\int_{n(\eta;0,0)}^{n(\eta;x,t)}\zeta g(\zeta;\eta)\mathrm{d}\zeta+\int\mathrm{d}\mu\frac{1}{\mu}\log\left|\frac{\eta+\mu}{\eta-\mu}\right|\int_{n(\mu;0,0)}^{n(\mu;x,t)}g(\zeta;\mu)\mathrm{d}\zeta\,,$$

where $g(\zeta;\eta)$ are functional degrees of freedom.

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