Introduction to Generalised Hydrodynamics in integrable field theories

Disordered Systems Advanced Lectures Series
2nd lecture

Thibault Bonnemain, 11th December 2023

Recap of the 1st lecture

- Large Hamiltonian systems generically relax to equilibrium at large time for a large class of interaction potentials.

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- At equilibrium, homogeneous system characterised by the Gibbs measure

$$\mu_{\mathrm{GE}} = \frac{1}{Z_{\mathrm{GE}}} \sum_{N=0}^{\infty} \exp\left[-\beta(E - \mu N - \nu P)\right] \frac{1}{N!} \mathrm{d}^N \mathbf{x} \mathrm{d}^N \mathbf{p} \ .$$

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- Inhomogeneous systems: propagation of local equilibria
  - Hydrodynamic approximation: $\langle o(x,t) \rangle \approx \langle o \rangle_{\{\beta_n(x,t)\}} \equiv \bar{o}_n(x,t) \ ,$
  - Macroscopic conservation laws: $\partial_t \bar{q}_n(x,t) + \partial_x \bar{j}_n(x,t) = 0 \ .$

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- Conservation laws: systems of hydrodynamic type (sometimes) solvable via hodograph method.

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- System: chain of $N = 64$ particles interacting with nearest neighbours

$$x_1 = x_N = 0 \; ,$$

$$H = \sum_{j=1}^{N} \left[\frac{p_j^2}{2} + V(x_{j+1} - x_j) \right] \; ,$$

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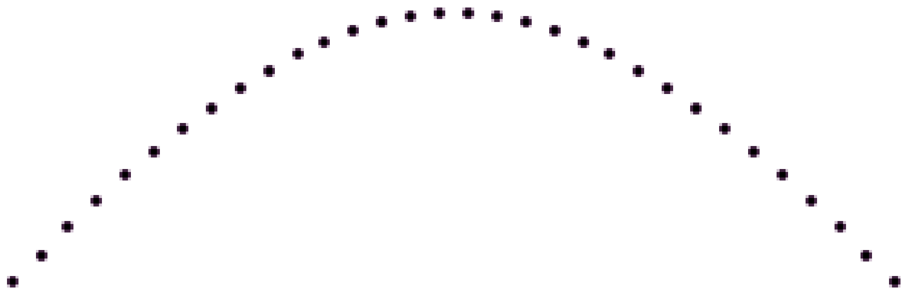
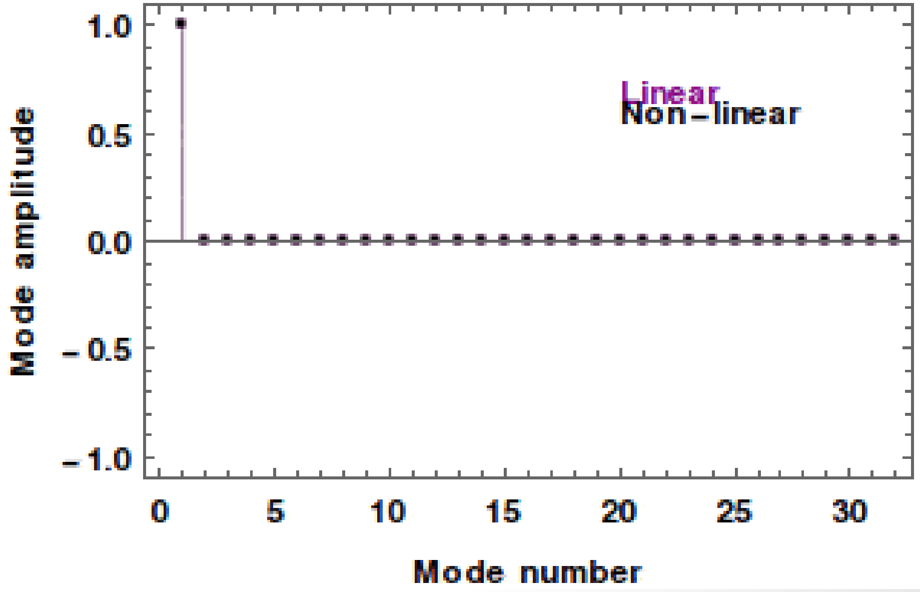
- Hypothesis: any Hamiltonian system with nonlinear forces should be ergodic

$$V(x) = \frac{\omega_0}{2} x^2 + \frac{\alpha}{3} x^3 \ .$$

Thermalisation and the « FPUT paradox »

- Initial condition: single sine wave.





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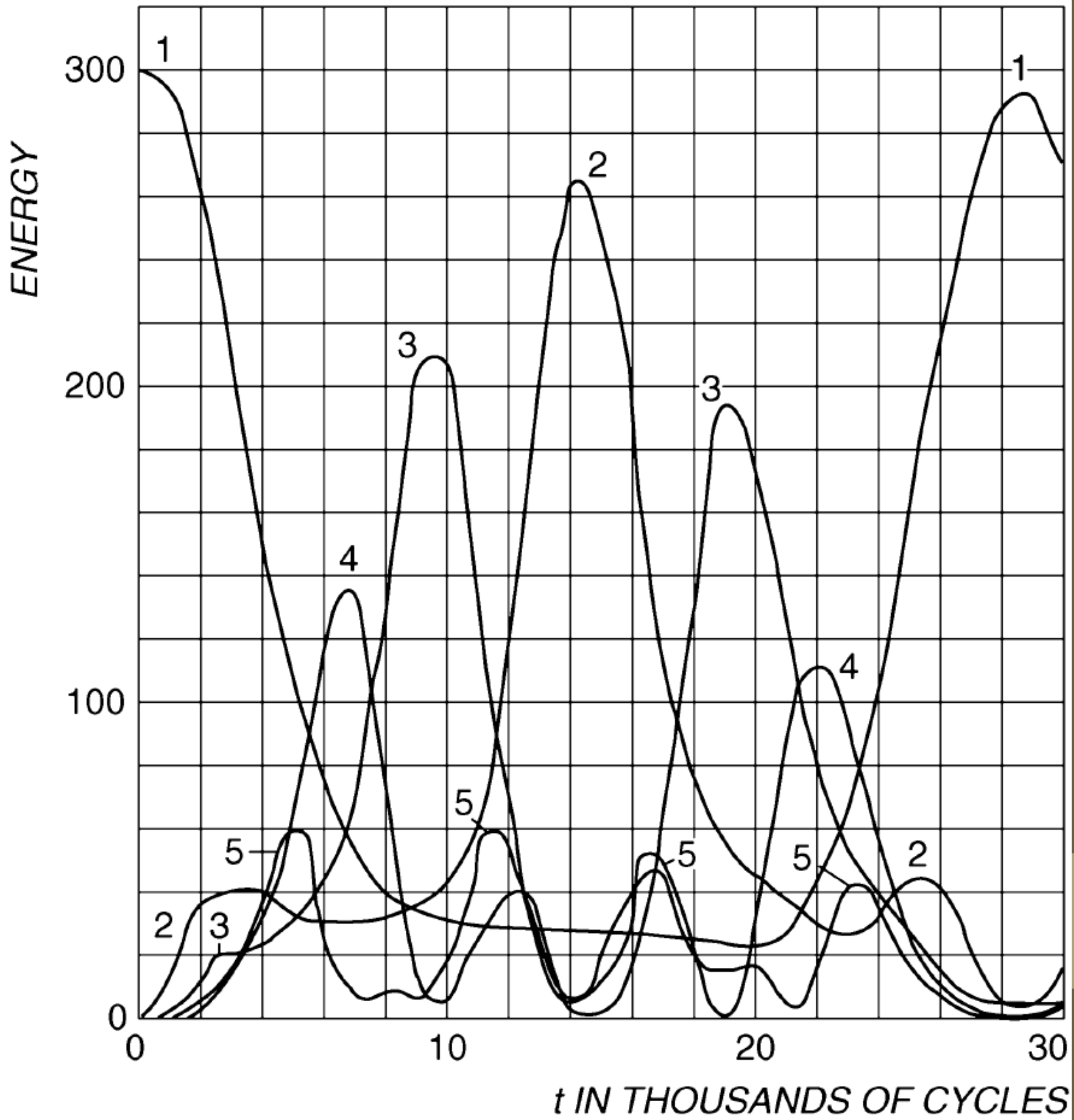

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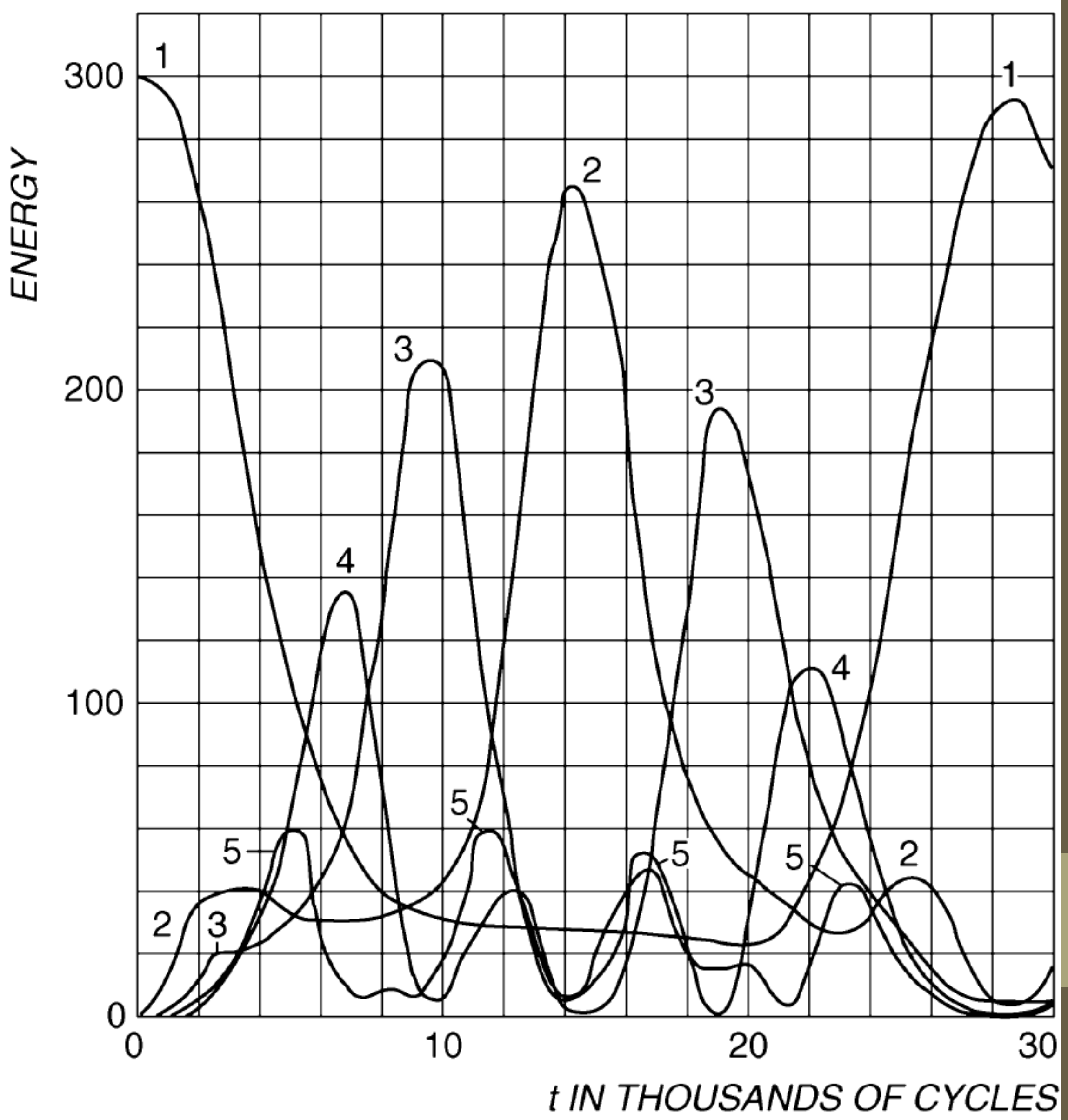

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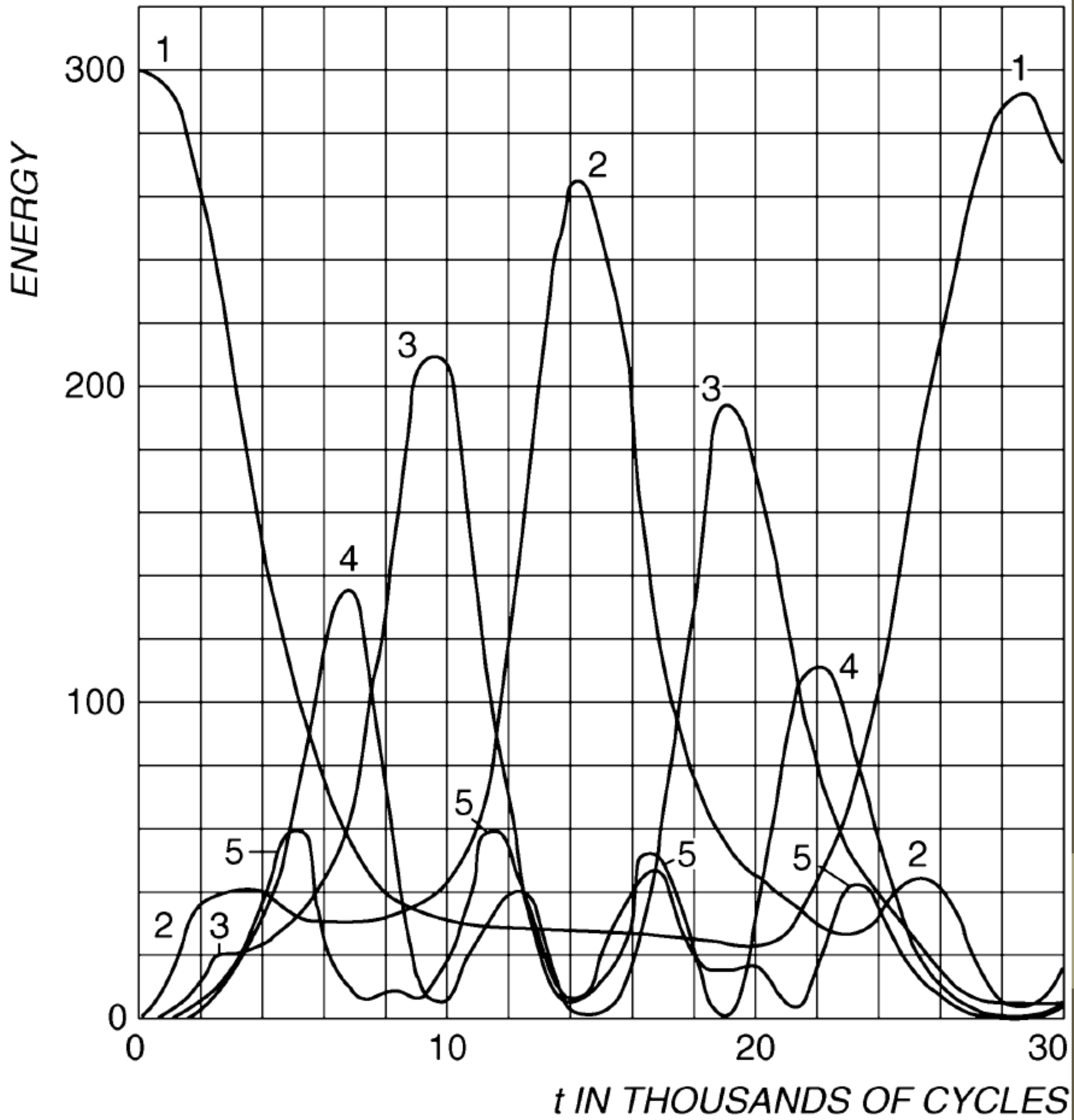

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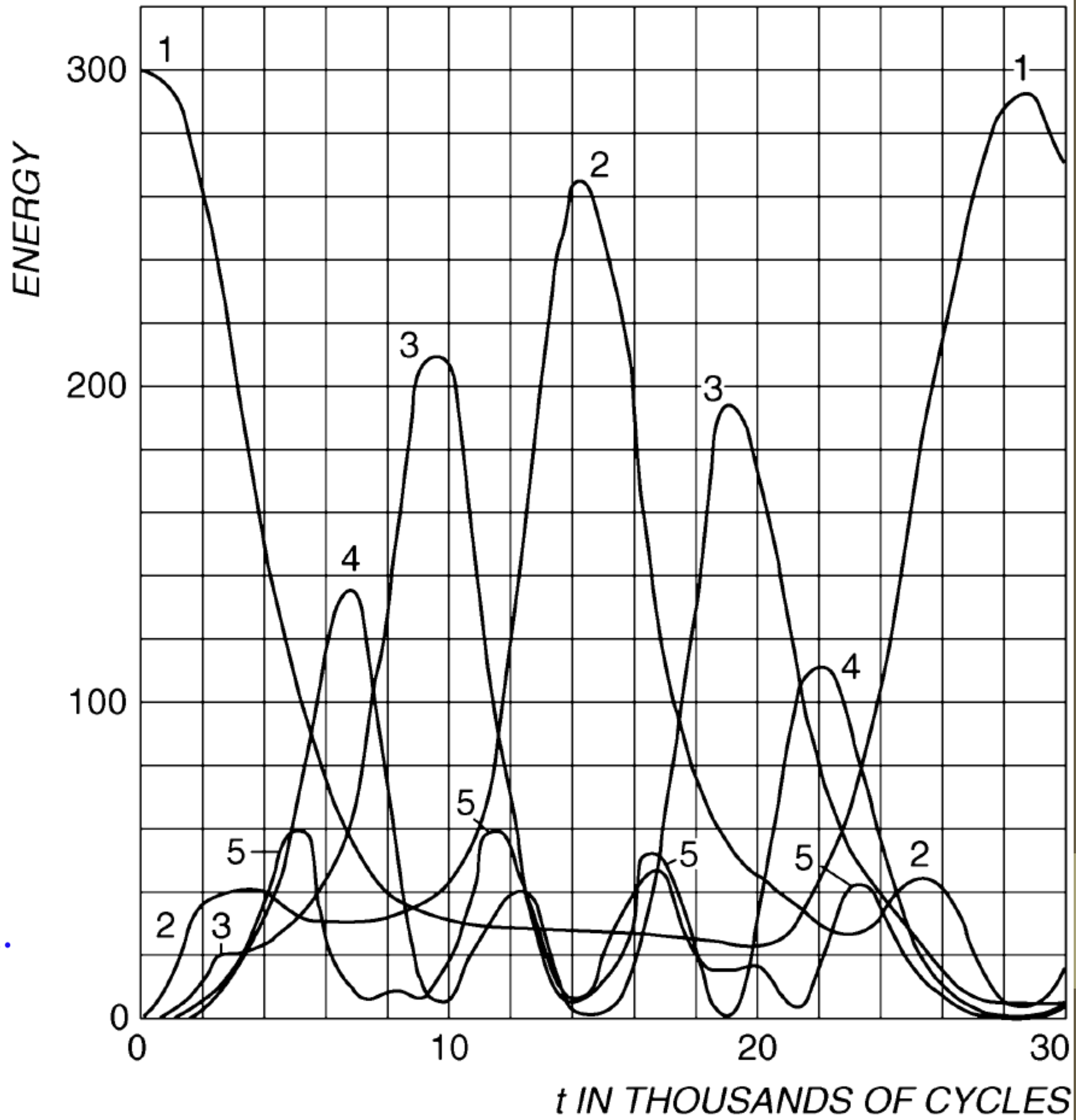
  Integrability impedes thermalisation.



Outline of the lectures

I. **Elements of Hydrodynamics**

II. **Integrable field theories**

1) From FPUT to KdV.
2) Integrability of KdV: Poisson structure, ZCR, Lax pair...
3) Inverse Scattering Transform: the cnoidal wave as an example.
4) The KdV $\tau-$function and $N-$soliton solutions.

III. **Soliton gas and Generalised Hydrodynamics**

IV. **Specific examples and potential extensions**

Continuum limit of α-FPUT

- Equations of motion for $\alpha-$FPUT

$$\ddot{q}_n = f(q_{n+1} - q_n) - f(q_n - q_{n-1}) \ ,$$

$$\text{with force} \quad f(Q) = \omega_0^2 Q + \alpha Q^2 \ .$$

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- Consider long waves: $\quad L \gg d$

$$\begin{aligned} q_{n\pm1} - q_n &= q(nd \pm d) - q(nd) \\ &= \pm d \ q_x + \frac{d^2}{2} \ q_{xx} \pm \frac{d^3}{6} \ q_{xxx} + \frac{d^4}{24} \ q_{xxxx} \pm \cdots \end{aligned}$$

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- Substition in the equations of motion

$$q_{tt} = \omega_0^2 \left[d^2 q_{xx} + \frac{d^4}{12} q_{xxxx} \right] + 2\alpha d^3 q_x q_{xx} + o(d^5) \ .$$

From Boussinesq to KdV

- Change of variables: $v = \frac{\alpha d}{6\omega_0^2} q_x, \quad T = \sqrt{24}\omega_0 t, \quad X = \frac{\sqrt{24}}{d} x,$

$$\boxed{v_{TT} = \left[v + 2v_{XX} + 6v^2\right]_{XX} \; .}$$

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$\sim \epsilon^4 \qquad \sim \epsilon^3 \qquad \sim \epsilon^3 \qquad \sim \epsilon^3$

Dispersion $\sim$ Nonlinearity $\Rightarrow$ $a = \frac{3}{2}$, and $b = \frac{1}{2}$.

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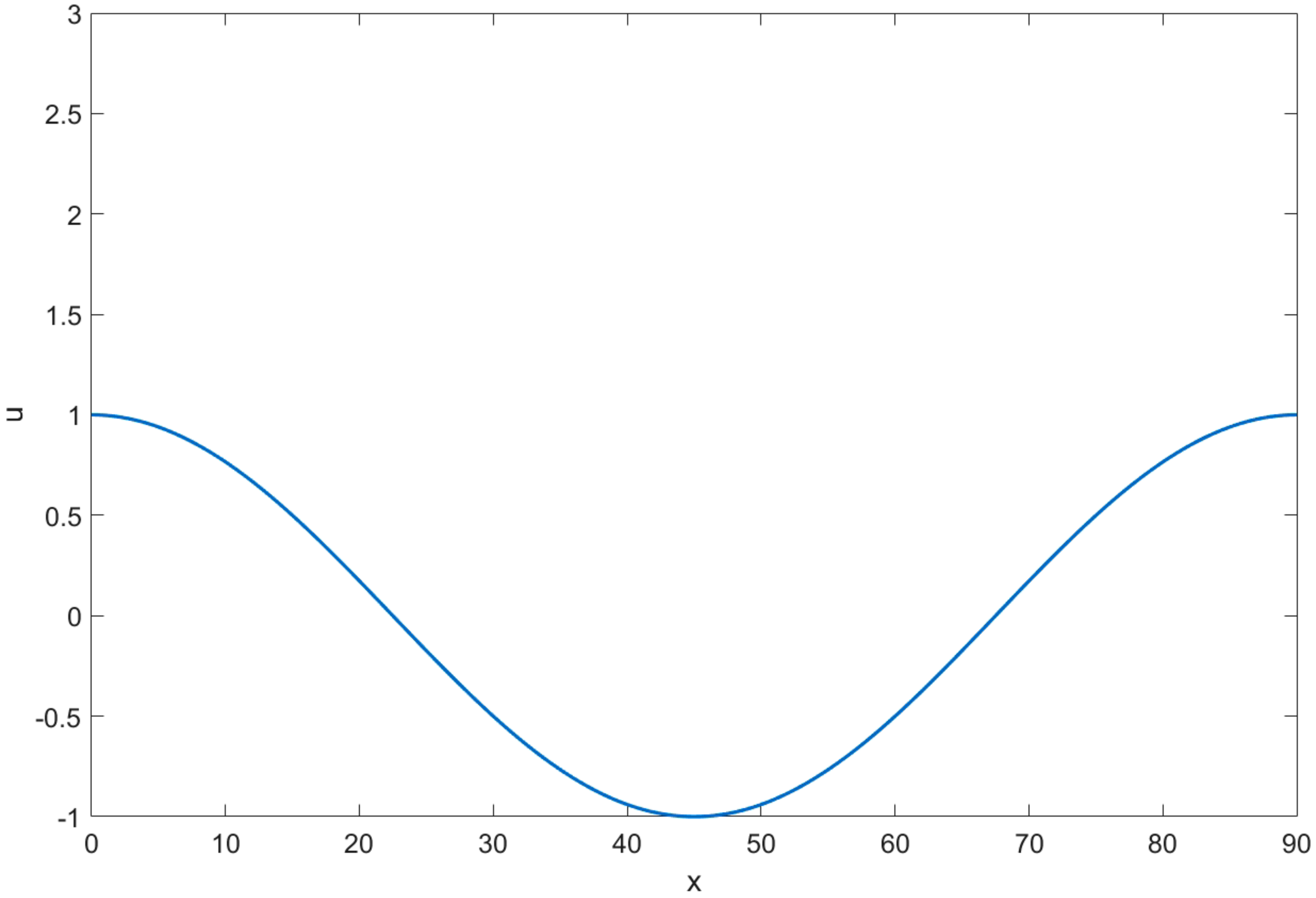
Dispersion $\sim$ Nonlinearity $\Rightarrow \quad a = \frac{3}{2}\,, \quad$ and $\quad b = \frac{1}{2}\ .$

- Change of variable: $v = \epsilon u + o(\epsilon^2)$

$$\boxed{u_\tau + 6uu_\xi + u_{\xi\xi\xi} = 0\ .}$$

Korteweg-de Vries equation

Zabusky and Kruskal 1965



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  - Has an infinite number of conserved quantities.
  - The system is bi-Hamiltonian.
  - There is a Zero-Curvature Representation (ZCR).
  - There is a Lax representation.
  - There is a $\tau-$function representation.
  - The system is solvable via the Inverse Scattering Transform (IST).
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⇒ KdV has all those properties!

Finite dimensional integrable Hamiltonian systems

- Phase space $M = \mathbb{R}^{2N}$ with Darboux coordinates

$$(\mathbf{q}, \mathbf{p}) = (q_1, \cdots, q_N; p_1, \cdots, p_n), \qquad \mathbf{X} = (\mathbf{q}, \mathbf{p})^T \ .$$

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$$J = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix} .$$

Symplectic form

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Symplectic form

- Poisson bracket: bilinear map for any two functions $f, \ g : M \to \mathbb{R}$

$$\{f; g\} = \frac{\partial f}{\partial \mathbf{X}} J \frac{\partial g}{\partial \mathbf{X}} = \sum_{i=1}^{N} \left[\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right] \ .$$

Finite dimensional integrable Hamiltonian systems

- Hamiltonian system characterised by $H : M \to \mathbb{R}$ yielding Hamilton's equations

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- Liouville integrability: there exists a set of N independent functions $\{f_i\}_{i=1}^{N}$

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Typically $f_1 = H$.

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- Liouville-Arnold theorem: $\mathbf{c} \in \mathbb{R}^N$, $M_{\mathbf{c}} = \{(\mathbf{q}; \mathbf{p}) \in M : f_i(\mathbf{q}; \mathbf{p}) = c_i, i = 1 \cdots N\}$

  - $M_{\mathbf{c}}$ smooth, invariant and diffeomorphic to $\quad T^N = S^1 \times \cdots \times S^1$.
  - Canonical transformation: $(\mathbf{q}; \mathbf{p}) \to (\mathbf{I}; \theta)$

$$\dot{\mathbf{I}} = 0 \ , \quad \text{and} \quad \dot{\theta} = \frac{\partial H}{\partial \mathbf{I}} = \text{const.} \ .$$

Action-angle coordinates

Infinite dimensional integrable Hamiltonian systems

- From finite to infinite dimensional systems

$$\mathbf{X}(t) \; \to \; u(x,t) \; , \qquad f(\mathbf{X}) \; \to \; \mathcal{F}[u] = \int_{\mathbb{R}} \mathrm{d}x \; F(x, u, u_x \cdots) \; ,$$

$$\{f;g\} = \frac{\partial f}{\partial \mathbf{X}} J \frac{\partial g}{\partial \mathbf{X}} \quad \to \quad \{\mathcal{F};\mathcal{G}\} = \int_{\mathbb{R}} \mathrm{d}x \; \frac{\delta \mathcal{F}}{\delta u(x)} \mathcal{J} \frac{\delta \mathcal{G}}{\delta u(x)} \; .$$

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- KdV is a bi-Hamiltonian system

$$u_t = \mathcal{J}_{.} \frac{\delta \mathcal{H}_{.}}{\delta u(x)}\;, \quad \text{with} \quad \left| \begin{array}{l} \mathcal{J}_1 = -\partial_{xxx} - 4u\partial_x - 2u_x \\ \mathcal{H}_1 = \displaystyle\int_{\mathbb{R}} \mathrm{d}x\; \frac{u^2}{2} \end{array} \right.\;, \quad \text{or} \quad \left| \begin{array}{l} \mathcal{J}_2 = \partial_x \\ \mathcal{H}_2 = \displaystyle\int_{\mathbb{R}} \mathrm{d}x\; \frac{u_x^2}{2} - u^3 \end{array} \right.\;.$$

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$$\Rightarrow \quad u_t = \mathcal{J}_1 \frac{\delta \mathcal{H}_1}{\delta u(x)} = \mathcal{J}_2 \frac{\delta \mathcal{H}_2}{\delta u(x)}\;, \quad \text{let} \quad \mathcal{J}_1 \frac{\delta \mathcal{H}_n}{\delta u(x)} = \mathcal{J}_2 \frac{\delta \mathcal{H}_{n+1}}{\delta u(x)} \quad \Rightarrow \quad \{\mathcal{H}_n;\mathcal{H}_m\} = 0\,.$$

Zero-curvature representation of KdV

- Consider the overdetermined linear system

$$\Psi_x = U\Psi \ , \qquad \Psi_t = V\Psi \ ,$$

$$U = \begin{pmatrix} 0 & 1 \\ -u-\lambda & 0 \end{pmatrix} , \qquad V = \begin{pmatrix} u_x & 4\lambda - 2u \\ 2u^2 + u_{xx} - 2u\lambda - 4\lambda^2 & -u_x \end{pmatrix} .$$

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- Compatibility conditions

$$\Psi_{xt} = \Psi_{tx} \quad \Leftrightarrow \quad U_t - V_x + [U;V] = \begin{pmatrix} 0 & 0 \\ u_t + 6uu_x + u_{xxx} & 0 \end{pmatrix} = 0 \ .$$

Zero-curvature representation of KdV

- Consider the overdetermined linear system

$$\Psi_x = U\Psi\ , \qquad \Psi_t = V\Psi\ ,$$

Lax connection

$$U = \begin{pmatrix} 0 & 1 \\ -u-\lambda & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} u_x & 4\lambda - 2u \\ 2u^2 + u_{xx} - 2u\lambda - 4\lambda^2 & -u_x \end{pmatrix}.$$

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- Geometric interpretation: Lax connection and parallel transport

$$\Omega_\gamma = \lim_{K\to\infty}\left[\mathcal{P}\prod_{k=1}^{K}\left(\mathbb{1} + \int_{\gamma_k}(U\mathrm{d}x + V\mathrm{d}t)\right)\right] \equiv \mathcal{P}\exp\int_\gamma (U\mathrm{d}x + V\mathrm{d}t)\ ,$$

γ curve in $\mathbb{R}^2$ from (x,t) to (y,s), $\gamma_1, \cdots, \gamma_K$ a partition into K adjacent segments.

Conservation laws from Zero-Curvature

- Introduce the transfer matrices

$$\mathcal{T}_t(x,y;\lambda) = \mathcal{P}\exp\int_x^y U(z,t;\lambda)\mathrm{d}z\ , \quad \mathcal{S}_x(s,t;\lambda) = \mathcal{P}\exp\int_s^t V(x,\tau;\lambda)\mathrm{d}\tau\ .$$

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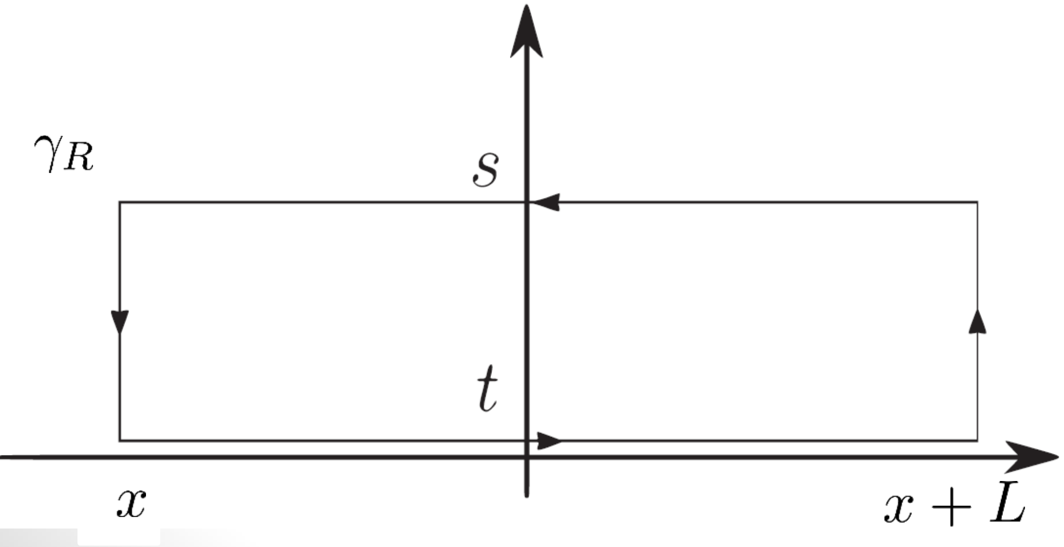
- Zero-curvature and non-Abelian Stokes theorem for rectangular loop γ_R

$$\Omega_{\gamma_R} = \mathcal{S}_x(t,s;\lambda)\mathcal{T}_t(x+L,x;\lambda)\mathcal{S}_{x+L}(s,t;\lambda)\mathcal{T}_s(x,x+L;\lambda) = \mathbb{1}$$



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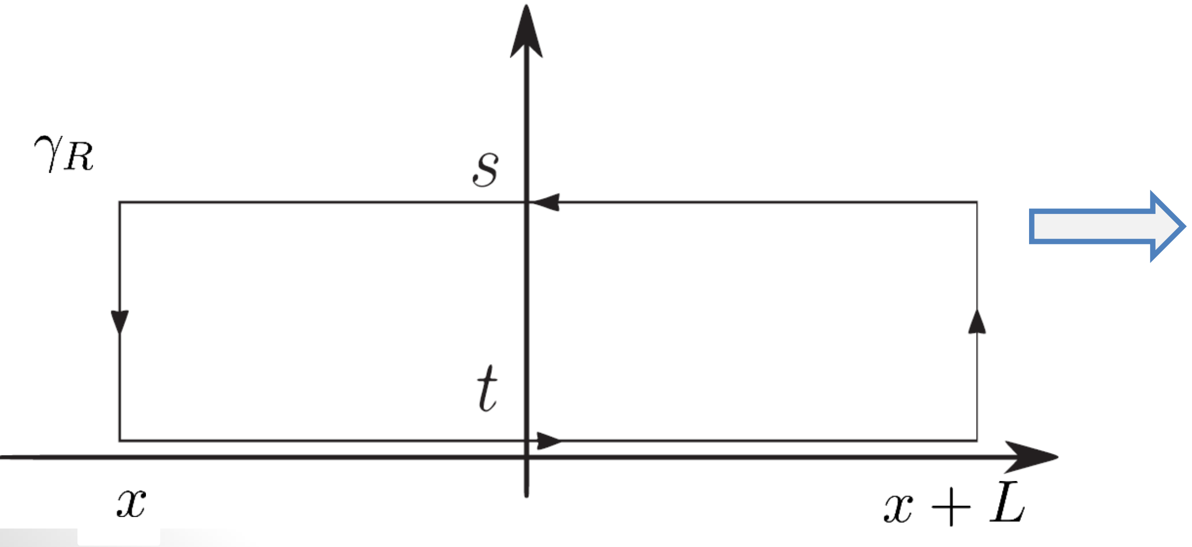
$$\Omega_{\gamma_R} = \mathcal{S}_x(t,s;\lambda)\mathcal{T}_t(x+L,x;\lambda)\mathcal{S}_{x+L}(s,t;\lambda)\mathcal{T}_s(x,x+L;\lambda) = \mathbb{1}$$



If $u(x,t)$ is periodic

$$\mathcal{S}_{x+L}(s,t;\lambda) = \mathcal{S}_x^{-1}(t,s;\lambda)\ ,$$

and

$$\mathrm{Tr}[\mathcal{T}_s(x,x+L;\lambda)] = \mathrm{Tr}[\mathcal{T}_t(x,x+L;\lambda)]\ .$$

Inverse Scattering Transform: rough sketch

- Use initial condition $u(x,t=0)$ to initialise the auxiliary linear problem

$$u(x,t=0) \quad \rightarrow \quad U(x,t=0;\lambda) \quad \rightarrow \quad \mathcal{T}_0(x,x+L;\lambda) \equiv \mathcal{T}_0(\lambda)\ .$$

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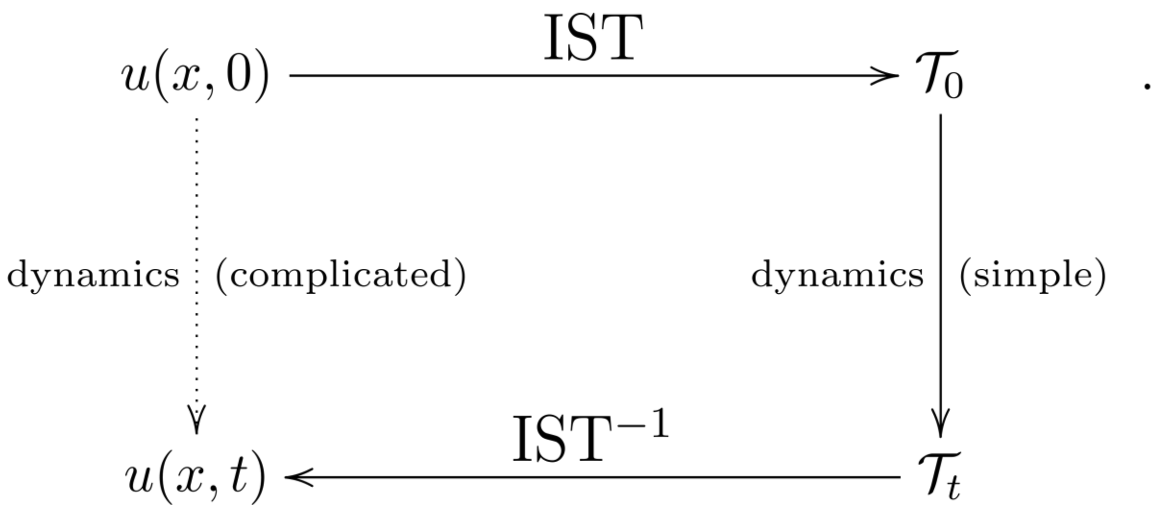
- Invert the IST: $\quad \mathcal{T}_t \quad \to \quad u(x, t)$



Lax representation of KdV

- Lax representation

$$\mathcal{L}\phi = \lambda\phi \ , \qquad \phi_t = \mathcal{M}\phi \ ,$$

$$\mathcal{L} = -\partial_{xx} - u(x,t) \ , \quad \mathcal{M} = u_x + [4\lambda - 2u(x,t)]\partial_x \ , \quad \dot{\mathcal{L}} = [\mathcal{M}; \mathcal{L}] \ .$$

Lax pair

Lax equation ⇔ KdV

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⇒ Solutions as linear combination of two basis solutions ϕ_- and ϕ_+

$$\phi_\pm(x_0) = 1 \ , \quad \phi'_\pm(x_0) = \pm ik \ , \quad k = \sqrt{\lambda} \ .$$

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Two remarks:

(i) If $u \in \mathbb{R}$ and $\lambda \in \mathbb{R}$, $\quad \phi_+ = \phi_-^*$.

(ii) If $\lambda \to \infty$, $\quad \phi_\pm \sim \exp[\pm ikx]$.

Band structure of the spectrum

- Periodicity of u implies invariance of $\mathcal{L} = \lambda\phi$ w.r.t. to translation $x \to x + L$

$$\begin{pmatrix} \phi_+(x+L) \\ \phi_-(x+L) \end{pmatrix} = T \begin{pmatrix} \phi_+(x) \\ \phi_-(x) \end{pmatrix}, \quad T = \begin{pmatrix} a(\lambda) & b(\lambda) \\ b^*(\lambda) & a^*(\lambda) \end{pmatrix}.$$

Monodromy operator

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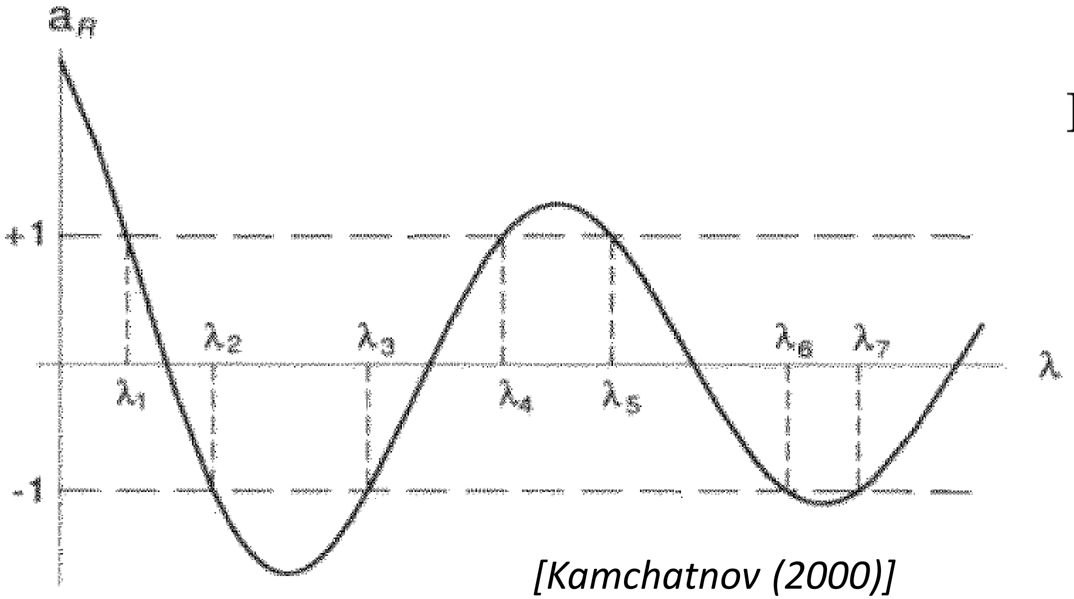

[Kamchatnov (2000)]

Eigenvalues : $\exp[\pm ip(\lambda)L] = a_R(\lambda) \pm \sqrt{a_R^2(\lambda) - 1}$.

Pseudo-momentum

⇩

$-1 < \cos[p(\lambda)L] = a_R(\lambda) < 1$!

Example of IST: the cnoidal wave

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- Introduce the squarred eigenfunction: $g(x,t;\lambda) = \phi_+(x,t;\lambda)\phi_-(x,t;\lambda)$

$$\begin{cases} g_t = (6ug + 2g_{xx} + 12g)_x \\ g_{xxx} + 2u_x g + 4(u+\lambda)g_x = 0 \end{cases}$$

Conservation equation

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$$\left.\begin{aligned} g = 1 + \frac{g_1}{\lambda} + \frac{g_2}{\lambda^2} + \cdots \\ \lambda g_x = -(g_{xxx} + 2u_x g + 4ug_x) \end{aligned}\right\} \Rightarrow \quad g_{n+1,x} = -(g_{n,xxx} + 2u_x g_n + 4ug_{n,x})$$

$$g_0 = 1 \qquad g_1 = -\frac{u}{2} \qquad g_2 = \frac{3}{8}u^2 + \frac{u_{xx}}{8} \qquad \text{etc.}$$

Example of IST: the cnoidal wave

- Integrating the second equation

$$\frac{gg_x}{2} - \frac{g_x^2}{4} + (\lambda + u)g^2 = R(\lambda) \ ,$$

with $R(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$.

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⇨ $u(x, t) = 2\mu(x, t) - (\lambda_1 + \lambda_2 + \lambda_3)$!

Example of IST: the cnoidal wave

- Setting the free spectral $\lambda = \mu(x,t)$ for all (x,t)

$$\begin{cases} \mu_x = \pm 2\sqrt{-R(\mu)} \\ \mu_t = (4\mu - 2u)\mu_x = -2(\lambda_1 + \lambda_2 + \lambda_3)\mu_x \end{cases}.$$

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Cnoidal wave

Cnoidal waves and solitons

- Wavelength of the cnoidal wave depends on size of the band: $0 < m = \frac{\lambda_3 - \lambda_2}{\lambda_3 - \lambda_1} < 1$

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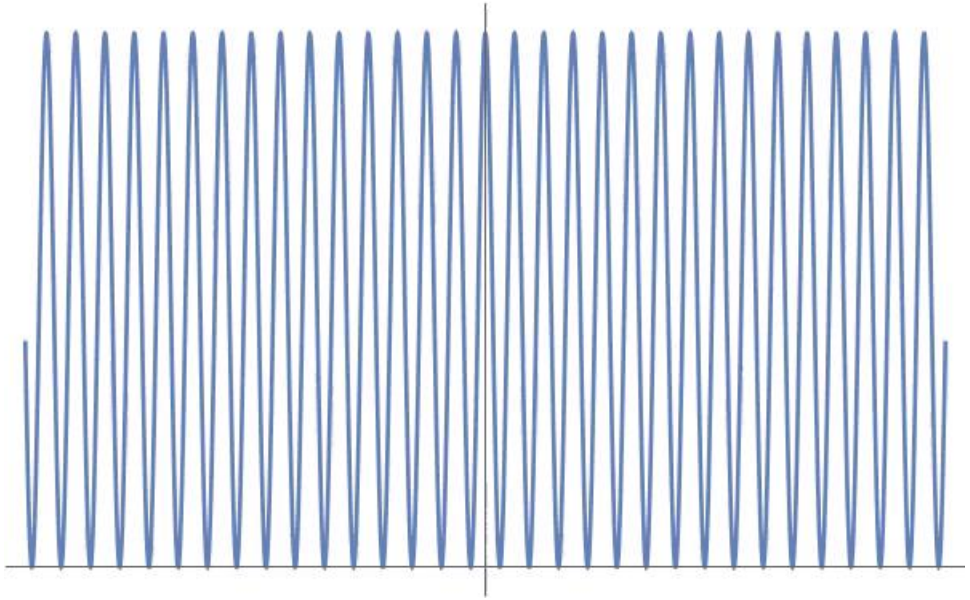

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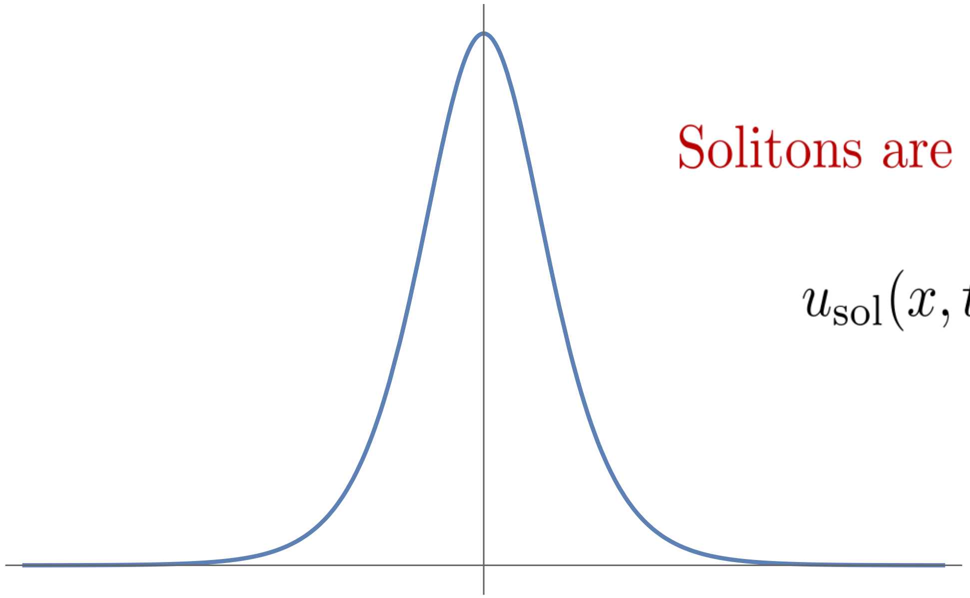

Solitons are cnoidal waves in the limit $\lambda_1 \to \lambda_2 \equiv \lambda_{12} \Rightarrow m \to 1$!

$$u_{\mathrm{sol}}(x,t) = -\lambda_3 \; + \; 2\eta^2 \mathrm{sech}^2\left[\eta(x - (4\eta^2 + 6\lambda_3)t + x_0)\right] \; ,$$

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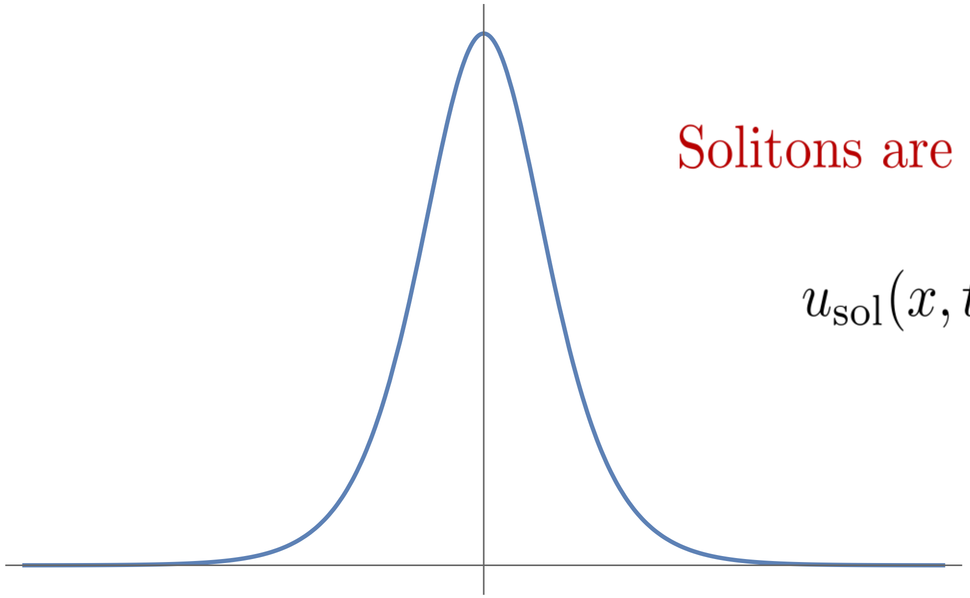

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- λ_3 plays the role of a background, by convention set $\lambda_3 = 0$

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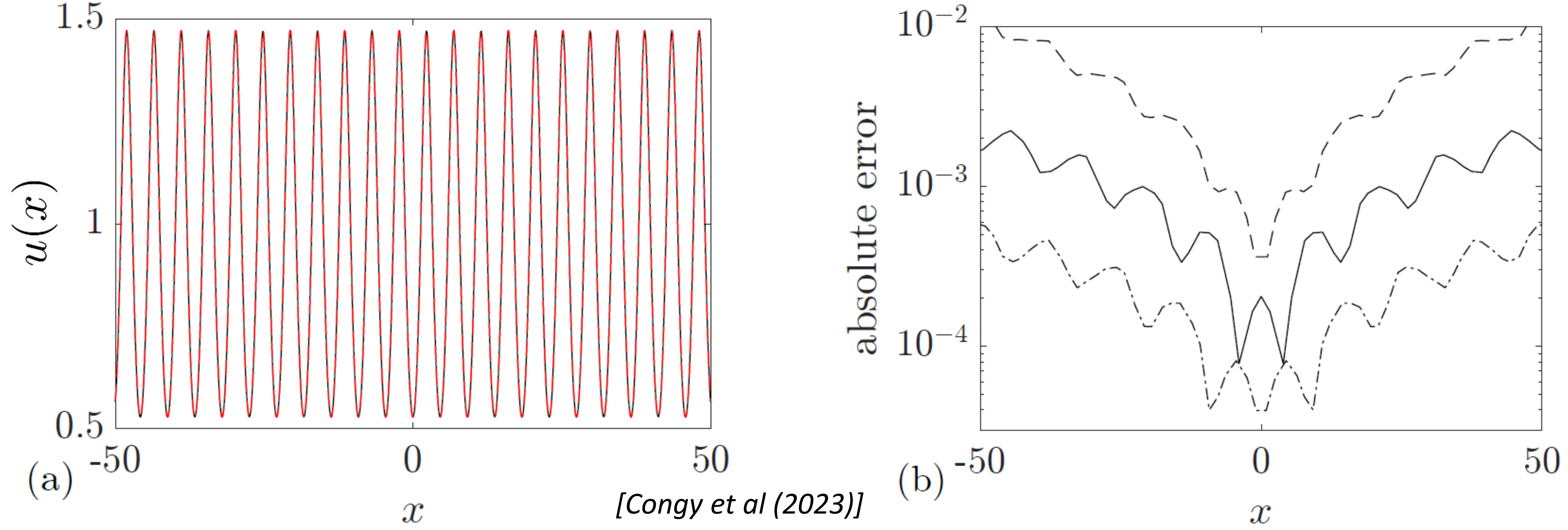
(ii) Solitons have particle-like behaviour: easier to make parallels with Stat Mech.

(iii) Any finite gap solution can be approximated by a $N-$soliton solution for N large enough.



[Congy et al (2023)]

τ-function formalism: Hirota's bilinear relations 1971

- Solution of KdV in terms of the τ−function

$$u_t + 6uu_x + u_{xxx} = 0 \ , \qquad u(x,t) = [\log \tau(x,t)]_{xx} \ .$$

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Hirota's D−operator $\quad D_s^n f \cdot g = (\partial_{s_1} - \partial_{s_2})^n f(s_1) g(s_2)|_{s_2=s_1=s} \ .$

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$$\tau(x,t) = 1 + \sum_{n=1}^{N} \sum_{{}_N C_n} a(i_1, i_2, \cdots, i_n) \exp\left[\theta_{i_1}(x,t) + \theta_{i_2}(x,t) + \cdots + \theta_{i_n}(x,t)\right]$$

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[Inspired by Kay, Moses (1956)]

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Interpretations of the τ-function

- $\tau-$function in terms of a Wronskian *[Satsuma (1979)]*

$$\tau(x,t) = \mathrm{Wr}(f_1, \ldots, f_N) = \det \begin{pmatrix} f_1 & f_2 & \cdots & f_N \\ f_1^{(1)} & f_2^{(1)} & \cdots & f_N^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(N-1)} & f_2^{(N-1)} & \cdots & f_N^{(N-1)} \end{pmatrix},$$

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- Sato theory: *[Sato (1981)]*
  - KdV is equivalent to the motion of a point on a Grassmanian manifold.
  - Hirota's bilinear equation is a Plücker relation.

- $\tau-$functions are partition functions in the spectral theory of random matrices.

[van Moerbeke (2000)]

The N-soliton τ-function as a determinant

- $\tau-$function as a determinant of a matrix

$$\tau = \det M \ , \quad M_{ij}(x,t) = \delta_{ij} + \frac{2\sqrt{\eta_i \eta_j}}{\eta_i + \eta_j} \exp\left[\theta_i(x,t) + \theta_j(x,t)\right] \ ,$$

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- Example: $1-$soliton solution

$$\begin{aligned} u_1(x,t) &= \log\left[\tau(x,t)\right]_{xx} \\ &= \log\left[1 + e^{2\theta_1(x,t)}\right]_{xx} \\ &= 2\eta_1^2 \operatorname{sech}^2\left[\eta_1\left(x - 4\eta_1^2 t - x_1^0\right)\right] \ . \end{aligned}$$

The N-soliton τ-function as a determinant

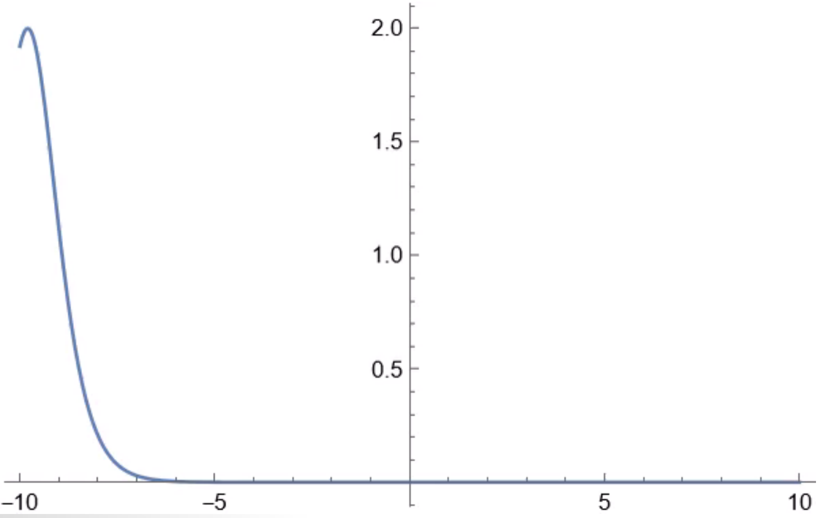
- $\tau-$function as a determinant of a matrix

$$\tau = \det M \ , \quad M_{ij}(x,t) = \delta_{ij} + \frac{2\sqrt{\eta_i \eta_j}}{\eta_i + \eta_j} \exp\left[\theta_i(x,t) + \theta_j(x,t)\right] \ ,$$

with $\theta_j(x,t) = \eta_j \left(x - 4\eta_j^2 t - x_j^0\right)$.

- Example: $1-$soliton solution

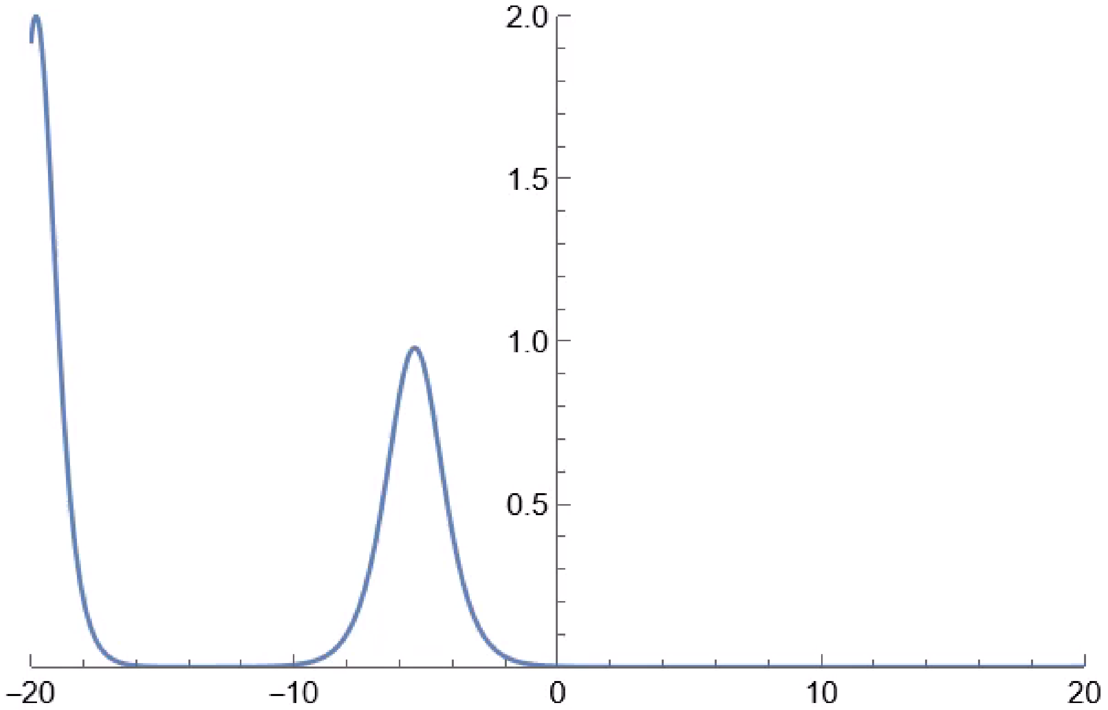
$$\begin{aligned} u_1(x,t) &= \log\left[\tau(x,t)\right]_{xx} \\ &= \log\left[1 + e^{2\theta_1(x,t)}\right]_{xx} \\ &= 2\eta_1^2 \operatorname{sech}^2\left[\eta_1\left(x - 4\eta_1^2 t - x_1^0\right)\right] \ . \end{aligned}$$



2-soliton solutions

- Example: 2−soliton solution

$$u_2(x,t) = \log\left[1 + e^{2\theta_1(x,t)} + e^{2\theta_2(x,t)} + \left(\frac{\eta_1 - \eta_2}{\eta_1 + \eta_2}\right)^2 e^{2(\theta_1(x,t)+\theta_2(x,t))}\right]_{xx} .$$



2-soliton solutions

$$\theta_j(x,t) = \eta_j\left(x - 4\eta_j^2 t + x_0\right)$$

- Example: 2−soliton solution

$$u_2(x,t) = \log\left[1 + e^{2\theta_1(x,t)} + e^{2\theta_2(x,t)} + \left(\frac{\eta_1 - \eta_2}{\eta_1 + \eta_2}\right)^2 e^{2(\theta_1(x,t)+\theta_2(x,t))}\right]_{xx} .$$

- Long time asymptotics, assuming $\eta_1 > \eta_2$

(i) $x \approx 4\eta_1^2 t$, $t \to -\infty \quad \Rightarrow \quad \theta_2 \to -\infty$ and θ_1 finite:

$$u_2(x,t) \approx \log\left[1 + e^{2\theta_1(x,t)}\right]_{xx} = 2\eta_1^2 \mathrm{sech}^2\left[\eta_1\left(x - 4\eta_1^2 t - x_1^0\right)\right] .$$

$$\varphi_{ij} = \log \left| \frac{\eta_i - \eta_j}{\eta_i + \eta_j} \right|$$

2-soliton solutions

$$\theta_j(x,t) = \eta_j \left(x - 4\eta_j^2 t + x_0 \right)$$

- Example: 2−soliton solution

$$u_2(x,t) = \log \left[1 + e^{2\theta_1(x,t)} + e^{2\theta_2(x,t)} + \left(\frac{\eta_1 - \eta_2}{\eta_1 + \eta_2} \right)^2 e^{2(\theta_1(x,t)+\theta_2(x,t))} \right]_{xx} .$$

- Long time asymptotics, assuming $\eta_1 > \eta_2$

(i) $x \approx 4\eta_1^2 t \ , \ t \to -\infty \quad \Rightarrow \quad \theta_2 \to -\infty$ and θ_1 finite:

$$u_2(x,t) \approx \log \left[1 + e^{2\theta_1(x,t)} \right]_{xx} = 2\eta_1^2 \mathrm{sech}^2 \left[\eta_1 \left(x - 4\eta_1^2 t - x_1^0 \right) \right] .$$

(ii) $x \approx 4\eta_1^2 t \ , \ t \to \infty \quad \Rightarrow \quad \theta_2 \to \infty$ and θ_1 finite:

$$u_2(x,t) \approx \log \left\{ e^{2\theta_2(x,t)} \left[1 + \left(\frac{\eta_1 - \eta_2}{\eta_1 + \eta_2} \right)^2 e^{2\theta_1(x,t)} \right] \right\}_{xx}$$
$$= 2\eta_1^2 \mathrm{sech}^2 \left[\eta_1 \left(x - 4\eta_1^2 t - x_1^0 \right) + \varphi_{12} \right]$$

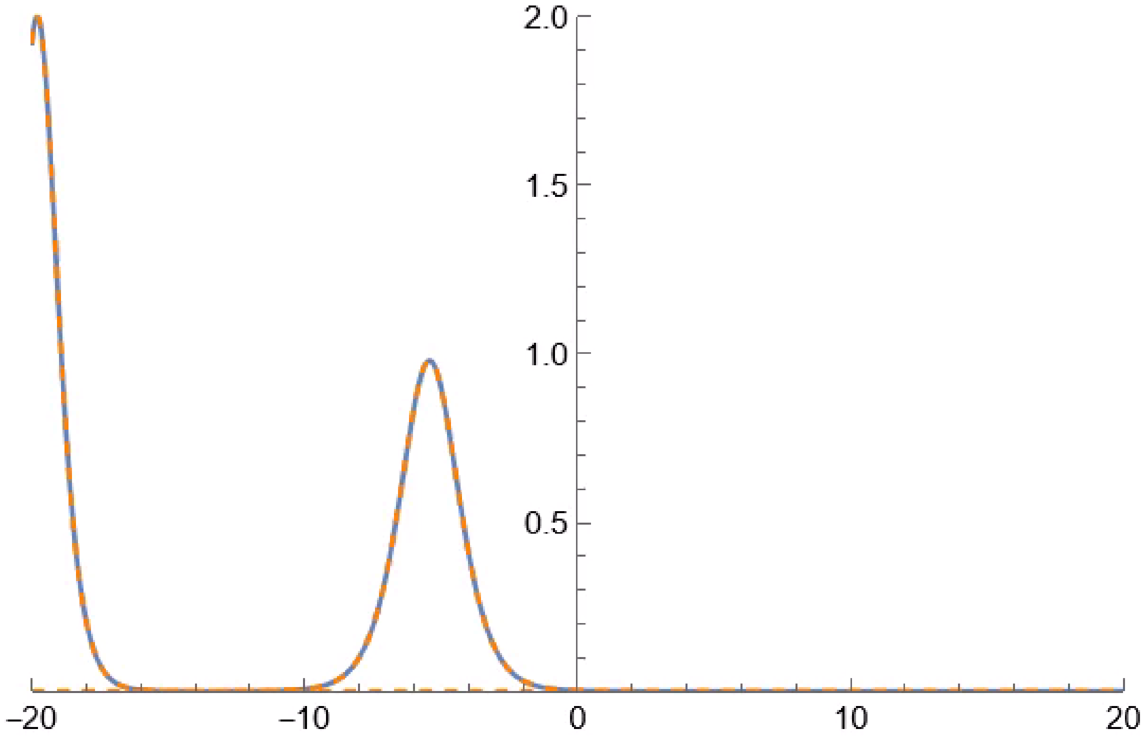
2-soliton solutions

$$\varphi_{ij} = \log\left|\frac{\eta_i - \eta_j}{\eta_i + \eta_j}\right|$$

$$\theta_j(x,t) = \eta_j\left(x - 4\eta_j^2 t + x_0\right)$$

- Example: 2−soliton solution

$$u_2(x,t) = \log\left[1 + e^{2\theta_1(x,t)} + e^{2\theta_2(x,t)} + \left(\frac{\eta_1 - \eta_2}{\eta_1 + \eta_2}\right)^2 e^{2(\theta_1(x,t)+\theta_2(x,t))}\right]_{xx}.$$



N-soliton solutions

- Long time asymptotics of N–soliton solutions

$$u_N(x,t) \approx \sum_{i=1}^{N} 2\eta_i^2 \operatorname{sech}^2\left[\eta_i\left(x - 4\eta_i^2 t - x_i^{\pm}\right)\right] \quad \text{as} \quad t \to \pm\infty.$$

N-soliton solutions

- Long time asymptotics of $N-$soliton solutions

$$u_N(x,t) \approx \sum_{i=1}^{N} 2\eta_i^2 \text{sech}^2 \left[\eta_i \left(x - 4\eta_i^2 t - x_i^{\pm}\right)\right] \quad \text{as} \quad t \to \pm\infty.$$

Action coordinate (η_i)

Angle coordinate ($4\eta_i^2 t - x_i^{\pm}$)

N-soliton solutions

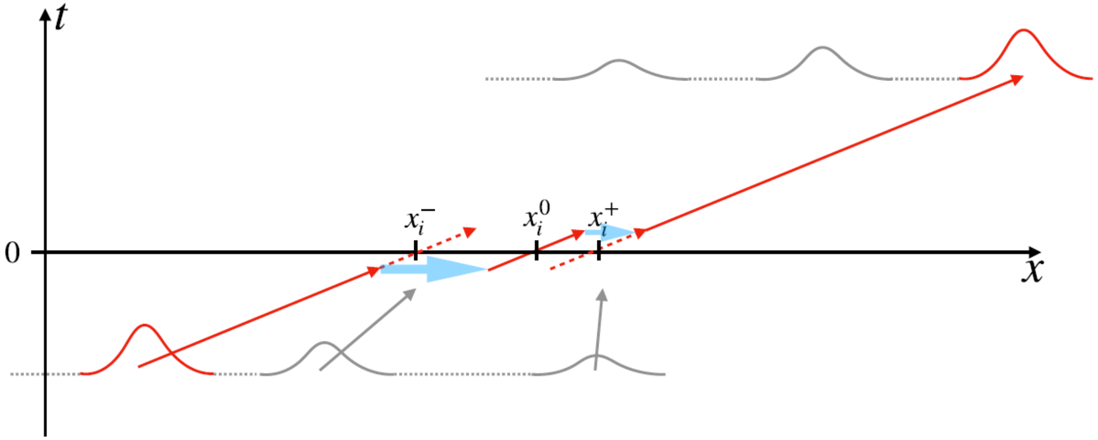
- Long time asymptotics of $N-$soliton solutions

$$u_N(x,t) \approx \sum_{i=1}^{N} 2\eta_i^2 \operatorname{sech}^2\left[\eta_i\left(x - 4\eta_i^2 t - x_i^{\pm}\right)\right] \quad \text{as} \quad t \to \pm\infty.$$

Action coordinate

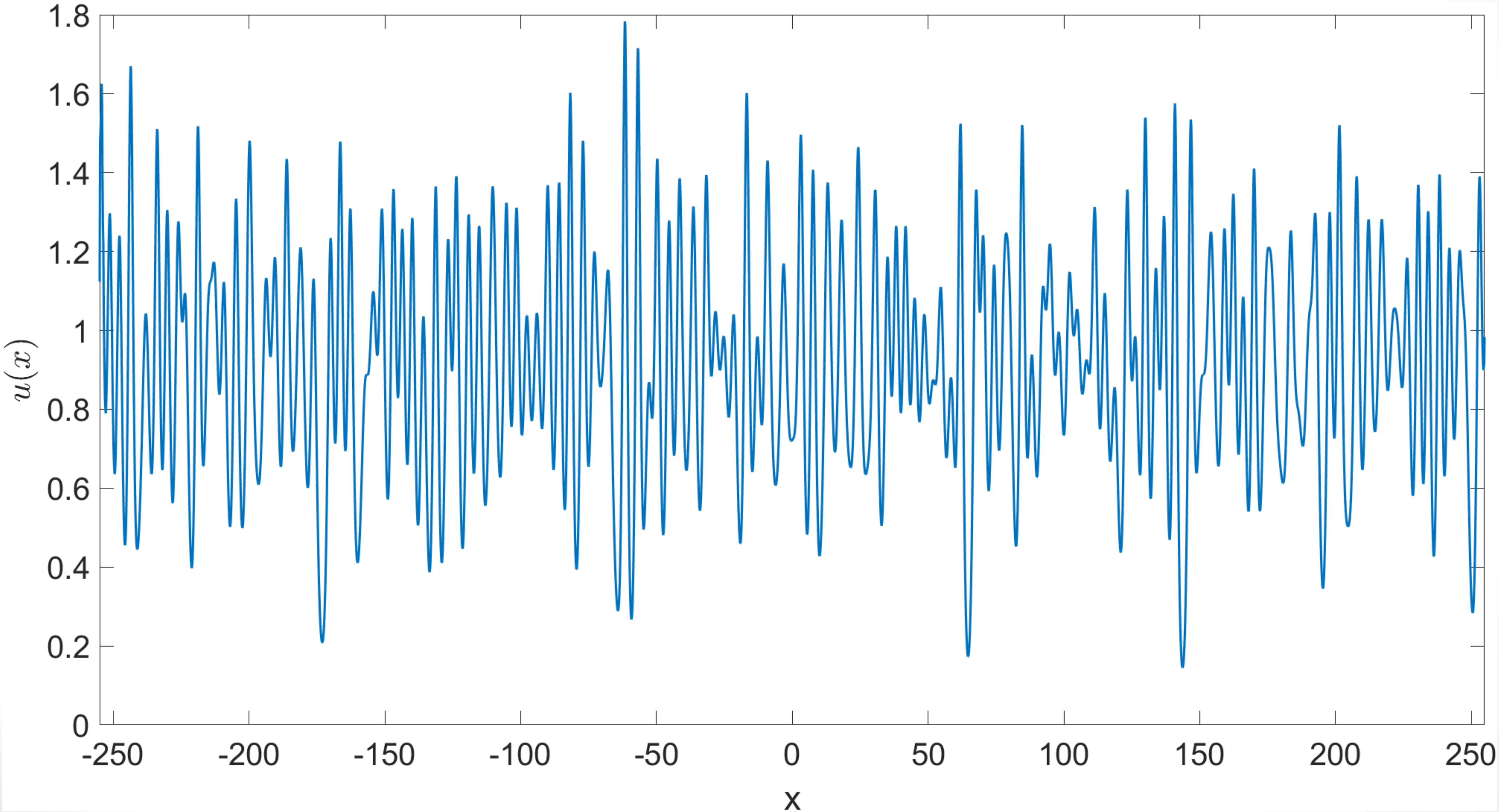
Angle coordinate

- Relation between asymptotic states given by scattering shift



$$x_i^+ - x_i^- = \sum_j \frac{\operatorname{sgn}(\eta_i - \eta_j)}{\eta_i} \ln\left|\frac{\eta_i + \eta_j}{\eta_i - \eta_j}\right| .$$

N-soliton solutions: example



References

Extensive review on FPUT

- G. Gallavotti, The Fermi-Pasta-Ulam problem: a status report, Springer (2007).

First observation of soliton fission and interaction

- N. J. Zabusky and M. D. Kruskal, Phys. Rev. Lett. 15(6), 240 (1965).

Derivation of KdV in various contexts and Inverse Scattering Transform

- M. J. Ablowitz, H. Segur, Solitons and the Inverse Scattering Transform, SIAM (1981).
- A Kamchatnov, Nonlinear periodic waves and their modulations: an introductory course, World Scientific (2000).

Hamiltonian formalism and Zero-curvature relation

Extensive discussion (but applied to NLS and not KdV)

- L. D. Faddeev and L. A. Takhtajan, Hamiltonian methods in the theory of solitons, Springer (1987).
- L. D. Faddeev and L. A. Takhtajan, Lett. Math. Phys. 10, 183 (1985). Poisson structure of KdV

τ-function

- R. Hirota, The direct method in soliton theory, Cambridge University Press (2004).
- R. Willox, J. Satsuma, Sato theory and transformation group: A unified approach to integrable systems, Springer (2004).
- P. van Moerbeke, Integrable lattices: random matrices and random permutations, arXiv (2000).