Introduction to Generalised Hydrodynamics in integrable field theories

Disordered Systems Advanced Lectures Series
1st lecutre

Thibault Bonnemain, 6th November 2023

Hydrodynamics in general

- Hydrodynamics is everywhere in physics:
    - Fluid dynamics (simple fluids Euler 1757)

Hydrodynamics in general

- Hydrodynamics is everywhere in physics:
  - Fluid dynamics (simple fluids Euler 1757)
  - Bose-Einstein condensates (two-fluid model)

Hydrodynamics in general

- Hydrodynamics is everywhere in physics:
  - Fluid dynamics (simple fluids Euler 1757)
  - Bose-Einstein condensates (two-fluid model)
  - Magneto hydrodynamics (plasma)

Hydrodynamics in general

- Hydrodynamics is everywhere in physics:
  - Fluid dynamics (simple fluids Euler 1757)
  - Bose-Einstein condensates (two-fluid model)
  - Magneto hydrodynamics (plasma)
  - Relativistic hydrodynamics (superdense neutron star)

Hydrodynamics in general

- Hydrodynamics is everywhere in physics:
  - Fluid dynamics (simple fluids Euler 1757)
  - Bose-Einstein condensates (two-fluid model)
  - Magneto hydrodynamics (plasma)
  - Relativistic hydrodynamics (superdense neutron star)
  - Mean-field game theory (crowd dynamics)

Hydrodynamics in general

- Hydrodynamics is everywhere in physics:
  - Fluid dynamics (simple fluids Euler 1757)
  - Bose-Einstein condensates (two-fluid model)
  - Magneto hydrodynamics (plasma)
  - Relativistic hydrodynamics (superdense neutron star)
  - Mean-field game theory (crowd dynamics)

$$\vdots$$

⇒ Generalised hydrodynamics (integrable systems)

[Castro-Alvaredo, Doyon, Yoshimura (2016)]

[Bertini, Collura, De Nardis, Fagotti (2016)]

Hydrodynamics in general

- Hydrodynamics is everywhere in physics:
  - Fluid dynamics (simple fluids Euler 1757)

$$\vdots$$

$\Rightarrow$ Generalised hydrodynamics (integrable systems)

[Castro-Alvaredo, Doyon, Yoshimura (2016)]

[Bertini, Collura, De Nardis, Fagotti (2016)]

- Derived from an underlying microscopic model:

$\Rightarrow$ field theories or many-particle systems

Hydrodynamics in general

- Hydrodynamics is everywhere in physics:
  - Fluid dynamics (simple fluids Euler 1757)

$$\vdots$$

$\Rightarrow$ Generalised hydrodynamics (integrable systems)

[Castro-Alvaredo, Doyon, Yoshimura (2016)]

[Bertini, Collura, De Nardis, Fagotti (2016)]

- Derived from an underlying microscopic model:

$\Rightarrow$ field theories or many-particle systems

- Main ingredients:

$\Rightarrow$ local conservation laws + propagation of local "equilibrium"

Outline of the lectures

I. **Elements of Hydrodynamics**

Outline of the lectures

I. **Elements of Hydrodynamics**

II. **Integrable field theories**

Outline of the lectures

I. **Elements of Hydrodynamics**

II. **Integrable field theories**

III. **Soliton gas and Generalised Hydrodynamics**

Outline of the lectures

I. **Elements of Hydrodynamics**
II. **Integrable field theories**
III. **Soliton gas and Generalised Hydrodynamics**
IV. **Specific examples and potential extensions**

Outline of the lectures

I. **Elements of Hydrodynamics**
   1) Propagation of local Equilibrium: from Statistical Mechanics to Hydrodynamics.
   2) Systems of hydrodynamic type.

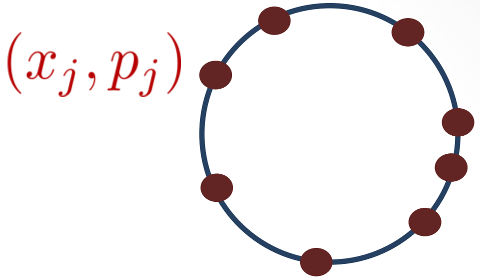
II. **Integrable field theories**

III. **Soliton gas and Generalised Hydrodynamics**

IV. **Specific examples and potential extensions**

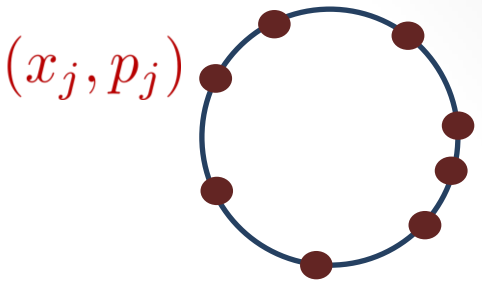
Back to basics: Newtonian dynamics

- N particles on a circle of perimeter L.



Back to basics: Newtonian dynamics

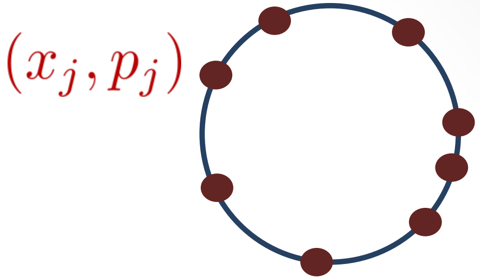
- N particles on a circle of perimeter L.
- Pair-wise interactions via short-ranged potential.



$$H_N = \sum_{j=1}^{N} \left[\frac{p_j^2}{2} + \sum_{i \neq j} V(|x_i - x_j|) \right]$$

$$\dot{x}_j(t) = p_j(t) \ , \qquad \dot{p}_j(t) = -\sum_{i \neq j} \partial_{x_j} V(|x_i - x_j|) \ .$$

Back to basics: Newtonian dynamics



- N particles on a circle of perimeter L.
- Pair-wise interactions via short-ranged potential.

$$H_N = \sum_{j=1}^{N} \left[\frac{p_j^2}{2} + \sum_{i \neq j} V(|x_i - x_j|) \right]$$

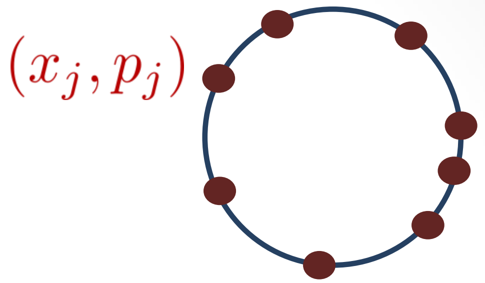
$$\dot{x}_j(t) = p_j(t) \ , \qquad \dot{p}_j(t) = -\sum_{i \neq j} \partial_{x_j} V(|x_i - x_j|) \ .$$

- Solutions generate a flow T_t on the phase space $\Gamma \equiv ([0, L] \times \mathbb{R})^N$ such that for

$$\bar{x}(t) \equiv \big(x_1(t), \cdots, x_N(t)\big) \ , \qquad \bar{p}(t) \equiv \big(p_1(t), \cdots, p_N(t)\big)$$

If $\big(\bar{x}(0), \bar{p}(0)\big) \in \Gamma$, $T_t\big(\bar{x}(0), \bar{p}(0)\big) = \big(\bar{x}(t), \bar{p}(t)\big)$.

Back to basics: Newtonian dynamics



- N particles on a circle of perimeter L.
- Pair-wise interactions via short-ranged potential.

$$H_N = \sum_{j=1}^{N} \left[\frac{p_j^2}{2} + \sum_{i \neq j} V(|x_i - x_j|) \right]$$

$$\dot{x}_j(t) = p_j(t) \ , \qquad \dot{p}_j(t) = -\sum_{i \neq j} \partial_{x_j} V(|x_i - x_j|)$$

- Solutions generate a flow T_t on the phase space $\Gamma \equiv ([0,L] \times \mathbb{R})^N$ such that if $\big(\bar{x}(0), \bar{p}(0)\big) \in \Gamma$, $T_t\big(\bar{x}(0), \bar{p}(0)\big) = \big(\bar{x}(t), \bar{p}(t)\big)$.
- Conservation laws

$$N \ , \qquad P = \sum_{j=1}^{N} p_j \ , \qquad E = H_N \ .$$

Issues with macroscopic systems

- Macroscopic description involves limits $N \to \infty$ and $L \to \infty$.

Issues with macroscopic systems

- Macroscopic description involves limits $N \to \infty$ and $L \to \infty$.
- Does the dynamics still exist?

Issues with macroscopic systems

- Macroscopic description involves limits $N \to \infty$ and $L \to \infty$.
- Does the dynamics still exist?
- Tractability of the dynamics?

Issues with macroscopic systems

- Macroscopic description involves limits $N \to \infty$ and $L \to \infty$.
- Does the dynamics still exist?
- Tractability of the dynamics?
- Imprecision of the initial data, deterministic chaos and randomness?

Issues with macroscopic systems

- Macroscopic description involves limits $N \to \infty$ and $L \to \infty$.
- Does the dynamics still exist?
- Tractability of the dynamics?
- Imprecision of the initial data, deterministic chaos and randomness?

Stochastic dynamics

Issues with macroscopic systems

- Macroscopic description involves limits $N \to \infty$ and $L \to \infty$.
- Does the dynamics still exist?
- Tractability of the dynamics?
- Imprecision of the initial data, deterministic chaos and randomness?

Stochastic dynamics VS Random initial data

Issues with macroscopic systems

- Macroscopic description involves limits $N \to \infty$ and $L \to \infty$.
- Does the dynamics still exist?
- Tractability of the dynamics?
- Imprecision of the initial data, deterministic chaos and randomness?

Stochastic dynamics VS Random initial data

Issues with macroscopic systems

- Macroscopic description involves limits $N \to \infty$ and $L \to \infty$.
- Does the dynamics still exist?
- Tractability of the dynamics?
- Imprecision of the initial data, deterministic chaos and randomness?

Stochastic dynamics VS Random initial data

$$\Downarrow$$

Statistical Mechanics as a tool to characterise "good" initial phase points.

Postulates of Statistical Mechanics

- States: probability measure of the phase space Γ.

Postulates of Statistical Mechanics

- States: probability measure of the phase space Γ.
- Initial measure μ_0 transported by T_t

$$\mu_0 = f\big(\bar{x}(0), \bar{p}(0)\big)\mathrm{d}^N\bar{x}\mathrm{d}^N\bar{p}\ , \quad \mu_t = f\big(\bar{x}(t), \bar{p}(t)\big)\mathrm{d}^N\bar{x}\mathrm{d}^N\bar{p}\ .$$

Postulates of Statistical Mechanics

- States: probability measure of the phase space Γ.
- Initial measure μ_0 transported by T_t

$$\mu_0 = f\big(\bar{x}(0), \bar{p}(0)\big)\mathrm{d}^N\bar{x}\mathrm{d}^N\bar{p}\ , \quad \mu_t = f\big(\bar{x}(t), \bar{p}(t)\big)\mathrm{d}^N\bar{x}\mathrm{d}^N\bar{p}\ .$$

- Boltzmann 1868: micro-canonical ensemble in long time limit

$$\mu_{\mathrm{mc}} = \lim_{\Delta E \to 0} \frac{1}{Z_{\mathrm{mc}}} \chi(\{E - \Delta E \leq H_N \leq E\}) \frac{1}{N!} \mathrm{d}^N\bar{x}\mathrm{d}^N\bar{p}\ .$$

Postulates of Statistical Mechanics

- States: probability measure of the phase space Γ.

- Initial measure μ_0 transported by T_t

$$\mu_0 = f\big(\bar{x}(0), \bar{p}(0)\big)\mathrm{d}^N\bar{x}\mathrm{d}^N\bar{p}\ , \quad \mu_t = f\big(\bar{x}(t), \bar{p}(t)\big)\mathrm{d}^N\bar{x}\mathrm{d}^N\bar{p}\ .$$

- Boltzmann 1868: micro-canonical ensemble in long time limit

$$\mu_{\mathrm{mc}} = \lim_{\Delta E \to 0} \frac{1}{Z_{\mathrm{mc}}} \chi(\{E - \Delta E \leq H_N \leq E\}) \frac{1}{N!} \mathrm{d}^N\bar{x}\mathrm{d}^N\bar{p}$$

$\widehat{=}$ Gibbs ensembles (GE) on $\Gamma = \cup_{N=0}^{\infty}\big(L \times \mathbb{R}\big)^N$:

$$\boxed{\mu_{\mathrm{GE}} = \frac{1}{Z_{\mathrm{GE}}} \sum_{N=0}^{\infty} \exp\left[-\beta(E - \mu N - \nu P)\right] \frac{1}{N!} \mathrm{d}^N\bar{x}\mathrm{d}^N\bar{p}}$$

Postulates of Statistical Mechanics

- States: probability measure of the phase space Γ.
- Initial measure μ_0 transported by T_t

$$\mu_0 = f\big(\bar{x}(0), \bar{p}(0)\big)\mathrm{d}^N\bar{x}\mathrm{d}^N\bar{p}\ , \quad \mu_t = f\big(\bar{x}(t), \bar{p}(t)\big)\mathrm{d}^N\bar{x}\mathrm{d}^N\bar{p}\ .$$

- Boltzmann 1868: micro-canonical ensemble in long time limit

$$\mu_{\mathrm{mc}} = \lim_{\Delta E \to 0} \frac{1}{Z_{\mathrm{mc}}} \chi(\{E - \Delta E \leq H_N \leq E\}) \frac{1}{N!} \mathrm{d}^N\bar{x}\mathrm{d}^N\bar{p}$$

$\widehat{=}$ Gibbs ensembles (GE) on $\Gamma = \cup_{N=0}^{\infty} \big(L \times \mathbb{R}\big)^N$:

Invariant !

$$\boxed{\mu_{\mathrm{GE}} = \frac{1}{Z_{\mathrm{GE}}} \sum_{N=0}^{\infty} \exp\left[-\beta(E - \mu N - \nu P)\right] \frac{1}{N!} \mathrm{d}^N\bar{x}\mathrm{d}^N\bar{p}}$$

[Gurevich et al. (1973)]

Physical observables and the infinite volume limit

- Physical observables: functions on the space of configurations.

Physical observables and the infinite volume limit

- Physical observables: functions on the space of configurations.

$$\Delta \subset [0,L]\,,\qquad \begin{aligned} n(\Delta) &= \sum_{j:x_j\in\Delta} 1 \;:\quad \text{number of particles in } \Delta \\ p(\Delta) &= \sum_{j:x_j\in\Delta} p_j \;:\quad \text{momentum of particles in } \Delta \\ e(\Delta) &= \sum_{j:x_j\in\Delta} \left[\frac{p_j^2}{2} + \sum_{i\neq j} V(|x_i - x_j|)\right] \;:\quad \text{energy of particles in } \Delta \end{aligned}$$

Physical observables and the infinite volume limit

- Physical observables: functions on the space of configurations.

$$n(\Delta) = \sum_{j:x_j\in\Delta} 1\,, \quad p(\Delta) = \sum_{j:x_j\in\Delta} p_j\,, \quad e(\Delta) = \sum_{j:x_j\in\Delta}\left[\frac{p_j^2}{2} + \sum_{i\neq j} V(|x_i - x_j|)\right]\,.$$

- Hydrodynamic fields

$$\mathsf{q}_0(x) = \sum_{j=1}^{N} \delta(x - x_j)$$

$$\mathsf{q}_1(x) = \sum_{j=1}^{N} \delta(x - x_j) p_j$$

$$\mathsf{q}_2(x) = \sum_{j=1}^{N} \delta(x - x_j)\left[\frac{p_j^2}{2} + \sum_{i\neq j} V(x_i - x_j)\right]$$

so that

$$n(\Delta) = \int_\Delta \mathrm{d}x\ \mathsf{q}_0(x)$$

$$p(\Delta) = \int_\Delta \mathrm{d}x\ \mathsf{q}_1(x)$$

$$e(\Delta) = \int_\Delta \mathrm{d}x\ \mathsf{q}_2(x)$$

Physical observables and the infinite volume limit

- Physical observables: functions on the space of configurations.

$$n(\Delta) = \sum_{j:x_j\in\Delta} 1\ , \quad p(\Delta) = \sum_{j:x_j\in\Delta} p_j\ , \quad e(\Delta) = \sum_{j:x_j\in\Delta}\left[\frac{p_j^2}{2} + \sum_{i\neq j} V(|x_i - x_j|)\right]\ .$$

- Hydrodynamic fields

$$n(\Delta) = \int_\Delta \mathrm{d}x\ \mathsf{q}_0(x)\ , \quad p(\Delta) = \int_\Delta \mathrm{d}x\ \mathsf{q}_1(x)\ , \quad e(\Delta) = \int_\Delta \mathrm{d}x\ \mathsf{q}_2(x)\ .$$

- Infinit volume limit

$$q_0 = \lim_{L\to\infty}\frac{1}{L}\langle n(L)\rangle(\beta,\mu,\nu)\ , \qquad q_1 = \lim_{L\to\infty}\frac{1}{L}\langle p(L)\rangle(\beta,\mu,\nu)$$

$$q_2 = \lim_{L\to\infty}\frac{1}{L}\langle e(L)\rangle(\beta,\mu,\nu)\ .$$

Correlation functions

- $g_n \; : \; ([0,L]\times\mathbb{R})^n \to \mathbb{R}$, continuous and of compact support, we define the corresponding $n-$particle observable

$$\Sigma(g_n) = \frac{1}{n!} \sum_{\substack{j_1,\cdots,j_n \\ j_1\neq\cdots\neq j_n}} g_n(x_{j_1}, p_{j_1}, \cdots, x_{j_n}, p_{j_n}) \; .$$

Correlation functions

- $g_n : ([0,L]\times\mathbb{R})^n \to \mathbb{R}$, continuous and of compact support, we define the corresponding $n-$particle observable

$$\Sigma(g_n) = \frac{1}{n!} \sum_{\substack{j_1,\cdots,j_n \\ j_1\neq\cdots\neq j_n}} g_n(x_{j_1}, p_{j_1}, \cdots, x_{j_n}, p_{j_n}) \ .$$

- Averages with respect to μ in terms of correlatin functions

$$\langle\Sigma(g_n)\rangle = \frac{1}{n!}\int \mathrm{d}^n\bar{x}\mathrm{d}^n\bar{p}\ C_n(x_1, p_1\cdots, x_n, p_n) g_n(x_1, p_1, \cdots, x_n, p_n) \ .$$

Correlation functions

- $g_n \; : \; ([0,L]\times\mathbb{R})^n \to \mathbb{R}$, continuous and of compact support, we define the corresponding $n-$particle observable

$$\Sigma(g_n) = \frac{1}{n!} \sum_{\substack{j_1,\cdots,j_n \\ j_1\neq\cdots\neq j_n}} g_n(x_{j_1},p_{j_1},\cdots,x_{j_n},p_{j_n}) \ .$$

- Averages with respect to μ in terms of correlatin functions

$$\langle\Sigma(g_n)\rangle = \frac{1}{n!}\int \mathrm{d}^n\bar{x}\mathrm{d}^n\bar{p}\; C_n(x_1,p_1\cdots,x_n,p_n)g_n(x_1,p_1,\cdots,x_n,p_n) \ .$$

- If μ_{GE} and V short-ranged, stable, 3 times continuously differentiable

$$\lim_{L\to\infty} C_{n,L}^{(\beta,\mu,\nu)}(x_1,p_1,\cdots x_n,p_n) = C_N^{(\beta,\mu,\nu)}(x_1,p_1,\cdots x_n,p_n) \ .$$

Correlation functions

- $g_n : ([0,L]\times\mathbb{R})^n \to \mathbb{R}$, continuous and of compact support, we define the corresponding $n-$particle observable

$$\Sigma(g_n) = \frac{1}{n!}\sum g_n(x_{j_1},p_{j_1},\cdots,x_{j_n},p_{j_n})\ .$$

- Averages with respect to μ in terms of correlatin functions

$$\langle\Sigma(g_n)\rangle = \frac{1}{n!}\int\ \mathrm{d}^n\bar{x}\mathrm{d}^n\bar{p}\ C_n(x_1,p_1\cdots,x_n,p_n)g_n(x_1,p_1,\cdots,x_n,p_n)\ .$$

- If μ_{GE} and V short-ranged, stable, 3 times continuously differentiable

$$\lim_{L\to\infty} C_{n,L}^{(\beta,\mu,\nu)}(x_1,p_1,\cdots x_n,p_n) = C_N^{(\beta,\mu,\nu)}(x_1,p_1,\cdots x_n,p_n)\ .$$

- Alternative approach: DLR consistency equations.

[Dubroshin (1968)]
[Lanford, Ruelle (1969)]

Local equilibrium and separation of scales

- Hydrodynamic fields

$$N = \int_0^L \mathrm{d}x \; \mathsf{q}_0(x) \; , \quad P = \int_0^L \mathrm{d}x \; \mathsf{q}_1(x) \; , \quad E = \int_0^L \mathrm{d}x \; \mathsf{q}_2(x) \; .$$

Local equilibrium and separation of scales

- Hydrodynamic fields

$$N = \int_0^L \mathrm{d}x\ \mathsf{q}_0(x)\ , \quad P = \int_0^L \mathrm{d}x\ \mathsf{q}_1(x)\ , \quad E = \int_0^L \mathrm{d}x\ \mathsf{q}_2(x)\ .$$

- Rewrite the equilibrium state

$$Z^{-1} \exp\left[-\sum_{n=0}^{2} \beta_n \int_0^L \mathrm{d}x\ \mathsf{q}_n(x)\right]$$

Local equilibrium and separation of scales

- Hydrodynamic fields

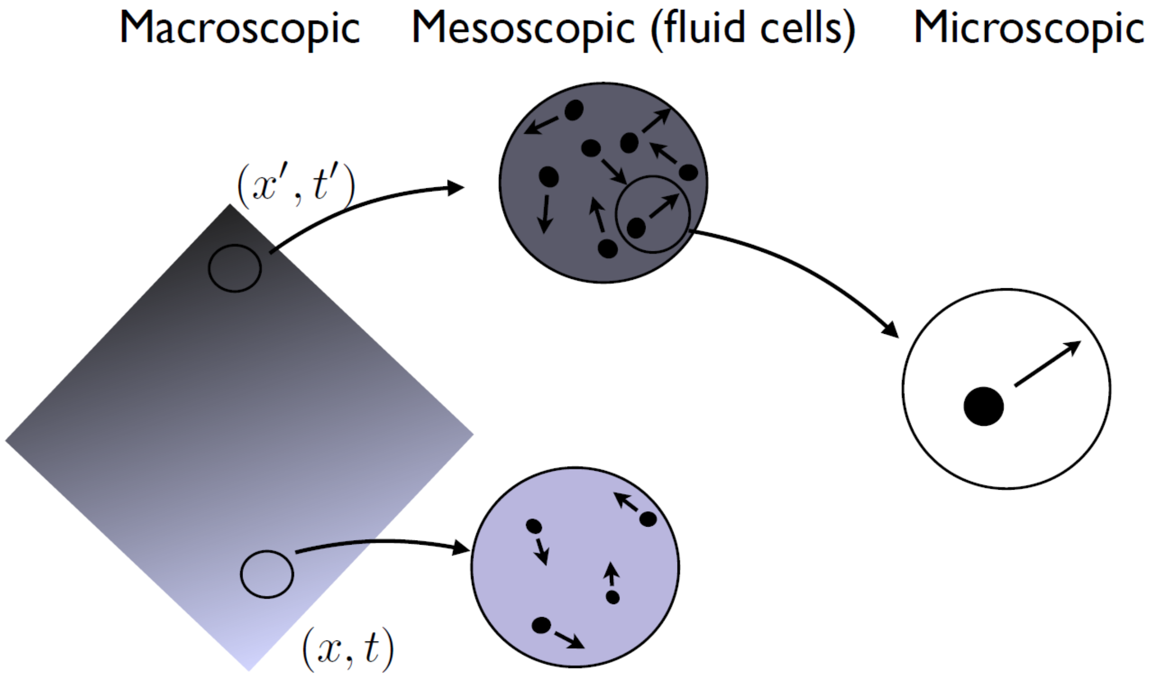
$$N = \int_0^L \mathrm{d}x \ \mathsf{q}_0(x) \ , \quad P = \int_0^L \mathrm{d}x \ \mathsf{q}_1(x) \ , \quad E = \int_0^L \mathrm{d}x \ \mathsf{q}_2(x) \ .$$

- Rewrite the equilibrium state

$$Z^{-1} \exp\left[-\sum_{n=0}^{2} \beta_n \int_0^L \mathrm{d}x \ \mathsf{q}_n(x) \right] \quad \to \quad Z^{-1} \exp\left[-\sum_{n=0}^{2} \int_0^L \mathrm{d}x \ \beta_n(x) \mathsf{q}_n(x) \right]$$

Local equilibrium and separation of scales

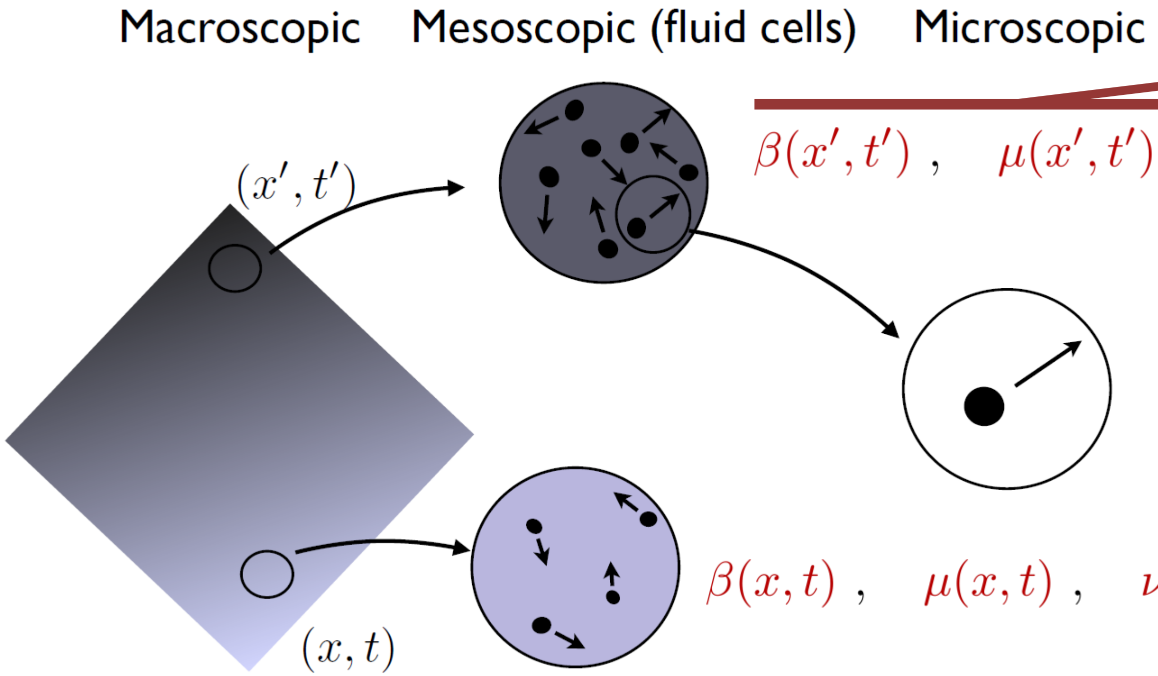
- Hydrodynamic fields: $\mathsf{q}_0(x)$, $\mathsf{q}_1(x)$, $\mathsf{q}_2(x)$.
- Rewrite the equilibrium state: $Z^{-1} \exp\left[-\sum_{n=0}^{2} \int_0^L \mathrm{d}x\ \beta_n(x)\mathsf{q}_n(x)\right]$.
- Separation of scales



[Doyon: Lecture Notes (2020)]

Local equilibrium and separation of scales

- Hydrodynamic fields: $\mathsf{q}_0(x)\ ,\quad \mathsf{q}_1(x)\ ,\quad \mathsf{q}_2(x)\ .$
- Rewrite the equilibrium state: $\quad Z^{-1}\exp\left[-\sum_{n=0}^{2}\int_0^L \mathrm{d}x\ \beta_n(x)\mathsf{q}_n(x)\right]\ .$
- Separation of scales



Slowly varying

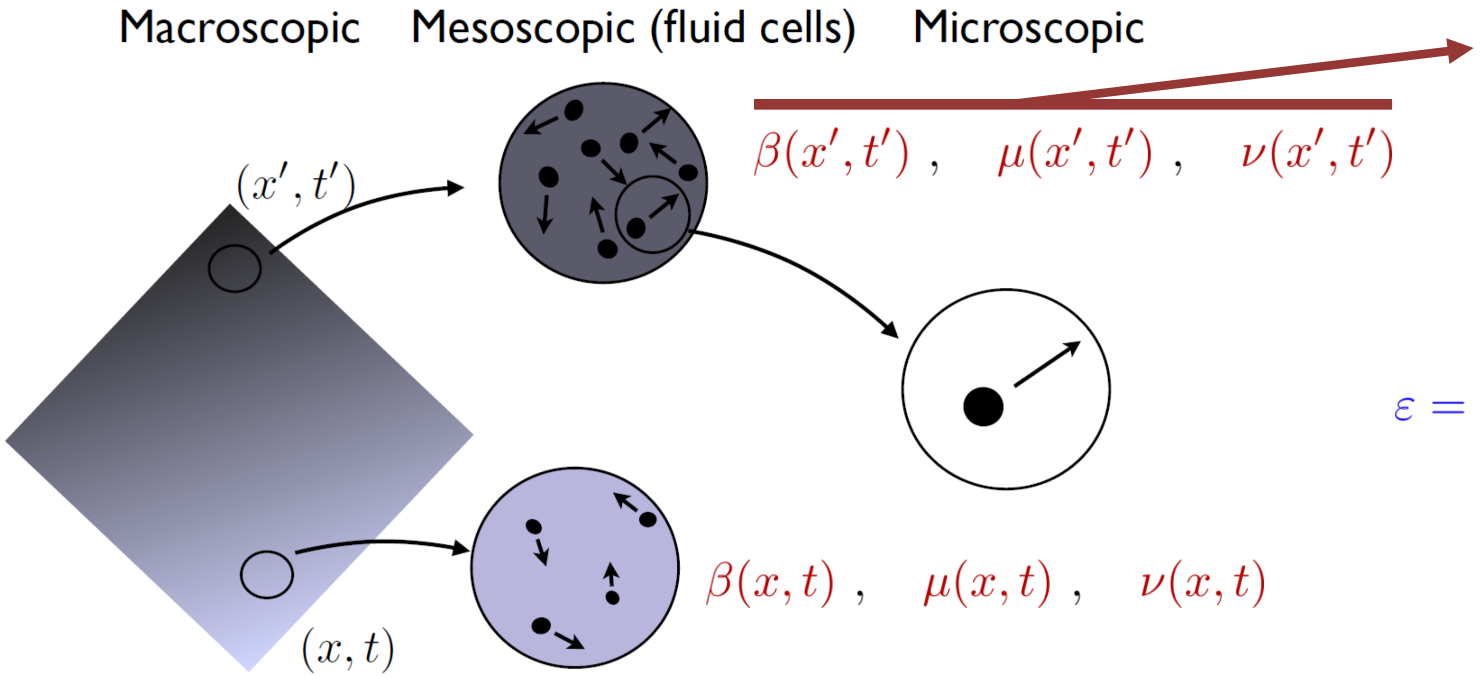
$\beta(x',t')\ ,\quad \mu(x',t')\ ,\quad \nu(x',t')$

$\beta(x,t)\ ,\quad \mu(x,t)\ ,\quad \nu(x,t)$

[Doyon: Lecture Notes (2020)]

Local equilibrium and separation of scales

- Hydrodynamic fields: $\mathsf{q}_0(x)\ ,\quad \mathsf{q}_1(x)\ ,\quad \mathsf{q}_2(x)\ .$
- Rewrite the equilibrium state: $\quad Z^{-1}\exp\left[-\sum_{n=0}^{2}\int_0^L \mathrm{d}x\ \beta_n(x)\mathsf{q}_n(x)\right]\ .$
- Separation of scales



Macroscopic Mesoscopic (fluid cells) Microscopic

Slowly varying

$\beta(x',t')\ ,\quad \mu(x',t')\ ,\quad \nu(x',t')$

$\beta(x,t)\ ,\quad \mu(x,t)\ ,\quad \nu(x,t)$

$$\varepsilon = \frac{\text{microscopic scale}}{\text{mesoscopic scale}}$$

[Doyon: Lecture Notes (2020)]

Scaling limit and law of large numbers

- Local equilibrium

$$Z_{\varepsilon}^{-1} \exp\left[-\sum_{n=0}^{2} \int_{0}^{L/\varepsilon} \mathrm{d}x \; \beta_n(\varepsilon x) \mathsf{q}_n(x) \right] \; .$$

Scaling limit and law of large numbers

- Local equilibrium

$$Z_{\varepsilon}^{-1} \exp \left[- \sum_{n=0}^{2} \int_{0}^{L/\varepsilon} \mathrm{d}x \; \beta_n(\varepsilon x) \mathsf{q}_n(x) \right] \, .$$

- Ensemble average of a local observable g spatially shifted by x/ε

$$\lim_{\varepsilon \to 0} \langle \tau_{x/\varepsilon} g \rangle_{\varepsilon} = \langle g \rangle \left(\beta(x), \mu(x), \nu(x) \right) \, .$$

Scaling limit and law of large numbers

- Local equilibrium

$$Z_{\varepsilon}^{-1} \exp\left[-\sum_{n=0}^{2} \int_{0}^{L/\varepsilon} \mathrm{d}x \; \beta_n(\varepsilon x)\mathsf{q}_n(x)\right] \; .$$

- Ensemble average of a local observable g spatially shifted by x/ε

$$\lim_{\varepsilon\to 0} \langle \tau_{x/\varepsilon} g\rangle_{\varepsilon} = \langle g\rangle \left(\beta(x), \mu(x), \nu(x)\right) \; .$$

- Law of large numbers ⇒ deterministic limit of spatial averages

$$\lim_{\varepsilon\to 0} \frac{1}{2l_{\varepsilon}} \int_{\frac{x}{\varepsilon}-l_{\varepsilon}}^{\frac{x}{\varepsilon}+l_{\varepsilon}} \mathrm{d}x' \; \tau_{x'/\varepsilon} g = \langle g\rangle \left(\beta(x), \mu(x), \nu(x)\right) \; ,$$

almost surely for l_{ε} such that $\lim_{\varepsilon\to 0} l_{\varepsilon} = \infty$, but $\lim_{\varepsilon\to 0} \varepsilon l_{\varepsilon} = 0$.

Fluctuations of order $\sim \varepsilon^{1/2}$

Scaling limit and law of large numbers

- Local equilibrium

$$Z_{\varepsilon}^{-1} \exp\left[-\sum_{n=0}^{2}\int_{0}^{L/\varepsilon} \mathrm{d}x \; \beta_n(\varepsilon x)\mathsf{q}_n(x)\right] \; .$$

- Ensemble average of a local observable g spatially shifted by x/ε

$$\lim_{\varepsilon\to 0}\langle\tau_{x/\varepsilon}g\rangle_{\varepsilon} = \langle g\rangle\left(\beta(x),\mu(x),\nu(x)\right) \; .$$

- Law of large numbers ⇒ deterministic limit of spatial averages

$$\lim_{\varepsilon\to 0}\frac{1}{2l_{\varepsilon}}\int_{\frac{x}{\varepsilon}-l_{\varepsilon}}^{\frac{x}{\varepsilon}+l_{\varepsilon}} \mathrm{d}x' \; \tau_{x'/\varepsilon}g = \langle g\rangle\left(\beta(x),\mu(x),\nu(x)\right) \; ,$$

almost surely for l_{ε} such that $\lim_{\varepsilon\to 0} l_{\varepsilon} = \infty$, but $\lim_{\varepsilon\to 0}\varepsilon l_{\varepsilon} = 0$.

Hydrodynamic approximation

$\times$

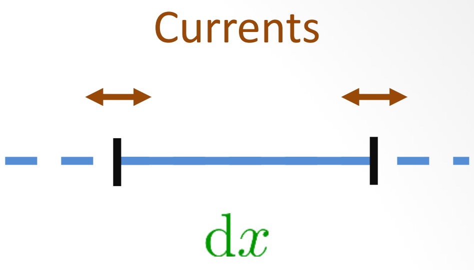
Ergodicity

Fluctuations of order $\sim \varepsilon^{1/2}$

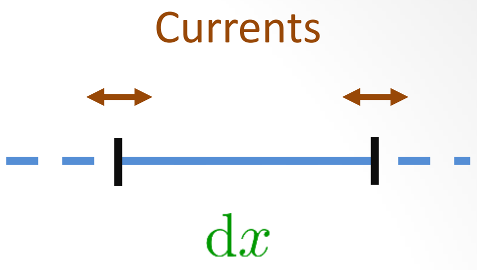
Hydrodynamic equations

- Local conservation laws

$$\partial_t \mathsf{q}_n(x,t) + \partial_x \mathsf{j}_n(x,t) = 0\ , \quad n = 0,\ 1,\ 2.$$



Hydrodynamic equations



- Local conservation laws

$$\partial_t \mathsf{q}_n(x,t) + \partial_x \mathsf{j}_n(x,t) = 0\ , \quad n = 0,\ 1,\ 2.$$

- Microscopic currents

$$\mathsf{j}_0(x) = \sum_{j=1}^{N} \delta(x - x_j) p_j = q_1(x)$$

$$\mathsf{j}_1(x) = \sum_{j=1}^{N} \delta(x - x_j) p_j^2 - \sum_{i \neq j} \partial_{x_j} V(|x_j - x_i|)(x_j - x_i) \int_0^1 \mathrm{d}\lambda \delta(\lambda x_j + (1-\lambda) x_i - x)$$

$$\mathsf{j}_2(x) = \sum_{j=1}^{N} \delta(x - x_j) p_j \left[\frac{p_j^2}{2} + \sum_{i \neq j} V(x_i - x_j) \right]$$

$$- \sum_{i \neq j} \left[(p_j + p_i) \partial_{x_j} V(|x_j - x_i|) \right] (x_j - x_i) \int_0^1 \mathrm{d}\lambda \delta(\lambda x_j + (1-\lambda) x_i - x)$$

Hydrodynamic equations

- Integrate local conservation laws over a fluid cell in space-time

$$\frac{1}{4l_\varepsilon\tau_\varepsilon}\int_{\frac{x}{\varepsilon}-l_\varepsilon}^{\frac{x}{\varepsilon}+l_\varepsilon}\mathrm{d}x\int_{\frac{t}{\varepsilon}-\tau_\varepsilon}^{\frac{t}{\varepsilon}+\tau_\varepsilon}\mathrm{d}t\,[\partial_t\mathsf{q}_n(x,t)+\partial_x\mathsf{j}_n(x,t)]=0\,,\quad n=0,\;1,\;2.$$

Hydrodynamic equations

- Integrate local conservation laws over a fluid cell in space-time

$$\frac{1}{4l_\varepsilon\tau_\varepsilon}\int_{\frac{x}{\varepsilon}-l_\varepsilon}^{\frac{x}{\varepsilon}+l_\varepsilon}\mathrm{d}x\int_{\frac{t}{\varepsilon}-\tau_\varepsilon}^{\frac{t}{\varepsilon}+\tau_\varepsilon}\mathrm{d}t\,[\partial_t\mathsf{q}_n(x,t)+\partial_x\mathsf{j}_n(x,t)]=0\;,\quad n=0,\;1,\;2.$$

- Hydrodynamic approximation (ergodicity and law of large numbers)

$$\begin{aligned}0\;&=\frac{1}{2\tau_\varepsilon}\left[\langle\mathsf{q}_n\rangle_{(\beta,\mu,\nu)}(x,t+\tau_\varepsilon)-\langle\mathsf{q}_n\rangle_{(\beta,\mu,\nu)}(x,t-\tau_\varepsilon)\right]\\&\quad+\frac{1}{2l_\varepsilon}\left[\langle\mathsf{j}_n\rangle_{(\beta,\mu,\nu)}(x+l_\varepsilon,t)-\langle\mathsf{j}_n\rangle_{(\beta,\mu,\nu)}(x-l_\varepsilon,t)\right]\end{aligned},$$

$$\text{w.r.t}\quad Z_\varepsilon^{-1}\exp\left[-\sum_{n=0}^{2}\int_0^{L/\varepsilon}\mathrm{d}x\beta_n(\varepsilon x,\varepsilon t)q_n(x)\right]+O(\varepsilon)\;.$$

Hydrodynamic equations

- Macroscopic conservation laws

$$\partial_t q_n(x,t) + \partial_x j_n(x,t) = 0 \ ,$$

$$q_n(x,t) = \langle \mathsf{q}_n \rangle_{(\beta,\mu,\nu)}(x,t) \ , \quad j_n(x,t) = \langle \mathsf{j}_n \rangle_{(\beta,\mu,\nu)}(x,t) \ .$$

Hydrodynamic equations

- Macroscopic conservation laws

$$\partial_t q_n(x,t) + \partial_x j_n(x,t) = 0 \ ,$$

$$q_n(x,t) = \langle \mathsf{q}_n \rangle_{(\beta,\mu,\nu)}(x,t) \ , \quad j_n(x,t) = \langle \mathsf{j}_n \rangle_{(\beta,\mu,\nu)}(x,t) \ .$$

- Euler equations

$$\begin{cases} \partial_t q_0 + \partial_x (v q_0) = 0 \\ \\ \partial_t (v q_0) + \partial_x (v^2 q_0 + p) = 0 \\ \\ \partial_t q_2 + \partial_x \left[v(q_2 + p) \right] = 0 \end{cases} \ ,$$

with

$$v q_0 = q_1 \ , \qquad p = \lim_{\varepsilon \to 0} \frac{\varepsilon}{L\beta(\varepsilon x, \varepsilon t)} \log(Z_\varepsilon) = p(q_0, q_2) \ .$$

Some references on statistical mechanics

Extensive review

- H. Spohn, Large scale dynamics of interacting particles, Springer (2012).

Gibbs measure

- R.L. Dubroshin, Theory Probability Appl. 13, 197-224 (1968).
- O.E. Lanford, D. Ruelle, Comm. Math. Phys. 13, 194-245 (1969).
- B.M. Gurevich, Y.G. Sinai, Y.M. Sukhov, Usp. Mat. Nauk. 28(5), 45-82 (1973).
- O.E. Lanford, Lecture notes in Physics 38, p1-111 (1975).

Hydrodynamics as a scaling limit

- A. Demasi et al., Nonequilibrium phenomena II From Stochastics to Hydrodynamics North Holland Phys. Pub., 123-294 (1984).
- S. Olla, S.R.S. Varadhan, H.T. Yau, Comm. Math. Phys. 155(3), 523-560 (1993).

Some historical context

- S.G. Brush, Arch. Hist. Exact Sci. 4, 145-183 (1967).
- C. Cercignani, Ludwig Boltzmann: The Man Who Trusted Atoms, Oxford University Press (2006).

Systems of hydrodynamic type

- Euler equations are an example of system of hydrodynamic type

$$\partial_t q_j + \partial_x f_j(q_0, \cdots, q_n) = 0 \ , \quad \text{for } j = 1,\ 2, \cdots,\ n\ .$$

Systems of hydrodynamic type

- Euler equations are an example of system of hydrodynamic type

$$\partial_t q_j + \partial_x f_j(q_0, \cdots, q_n) = 0 \ , \quad \text{for } j = 1,\ 2, \cdots,\ n \ .$$

- Quasilinear form, let $\mathbf{q} = (q_1, \cdots, q_n)$

$$\partial_t \mathbf{q} + A(\mathbf{q}) \partial_x \mathbf{q} = 0 \ ,$$

with the Jacobian matrix

$$A(\mathbf{q}) = \begin{pmatrix} \partial_{q_1} f_1 & \cdots & \partial_{q_n} f_1 \\ & \cdots & \\ \partial_{q_1} f_n & \cdots & \partial_{q_n} f_n \end{pmatrix} \ .$$

Systems of hydrodynamic type

- Euler equations are an example of system of hydrodynamic type

$$\partial_t q_j + \partial_x f_j(q_0, \cdots, q_n) = 0 \ , \quad \text{for } j = 1,\ 2, \cdots,\ n \ .$$

- Quasilinear form, let $\mathbf{q} = (q_1, \cdots, q_n)$

$$\partial_t \mathbf{q} + A(\mathbf{q}) \partial_x \mathbf{q} = 0 \ ,$$

with the Jacobian matrix

$$A(\mathbf{q}) = \begin{pmatrix} \partial_{q_1} f_1 & \cdots & \partial_{q_n} f_1 \\ & \cdots & \\ \partial_{q_1} f_n & \cdots & \partial_{q_n} f_n \end{pmatrix} \ .$$

- Strictly hyperbolic system if, for every $\mathbf{q}$, $A(\mathbf{q})$ has n real, distinct eigenvalues

$$v_1(\mathbf{q}) < \cdots < v_n(\mathbf{q}) \ .$$

Appetizer: linear systems

- Single linear conservation law

$$\partial_t q + v \partial_x q = 0$$

$$q(x,0) = \bar{q}(x)$$

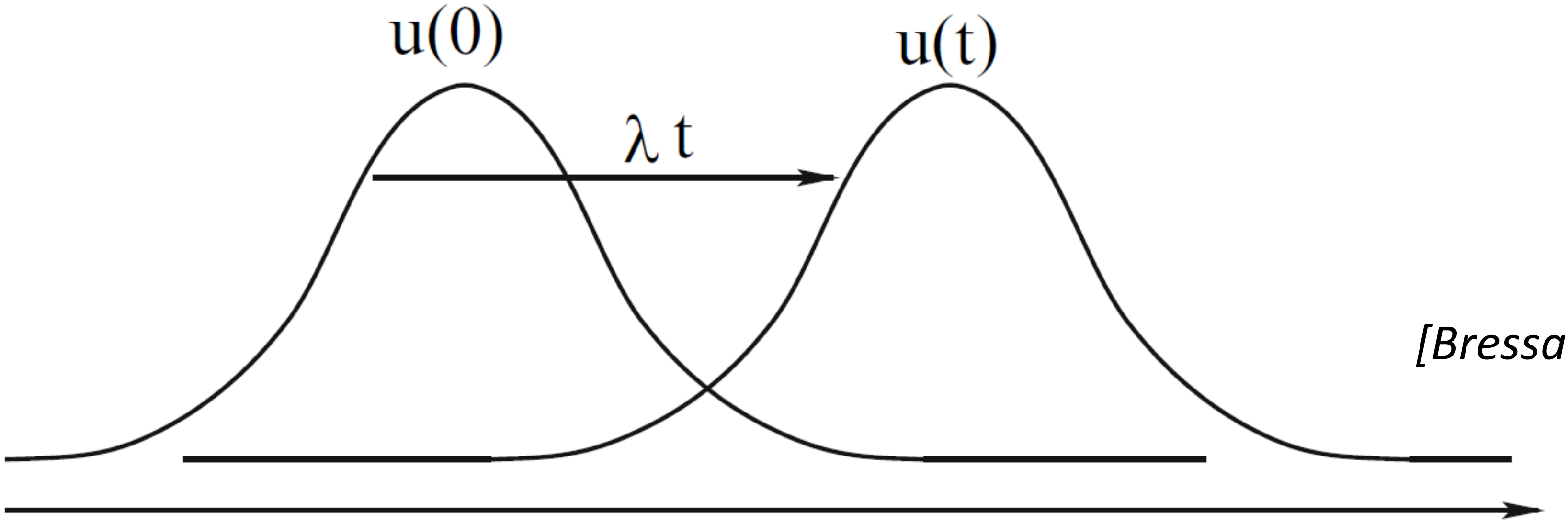
Appetizer: linear systems

- Single linear conservation law

$$\partial_t q + v\partial_x q = 0$$
$$q(x,0) = \bar{q}(x)$$

$$\Downarrow$$

$$q(x,t) = \bar{q}(x - vt)$$



[Bressan (2009)]

Appetizer: linear systems

- n conservation laws: A is a $n \times n$ matrix with real eigenvalues $v_1 < \cdots < v_n$

$$\partial_t \mathbf{q} + A \partial_x \mathbf{q} = 0 \ , \quad \mathbf{q}(x,0) = \bar{\mathbf{q}}(x) \ .$$

Appetizer: linear systems

- n conservation laws: A is a $n \times n$ matrix with real eigenvalues $v_1 < \cdots < v_n$

$$\partial_t \mathbf{q} + A \partial_x \mathbf{q} = 0 \ , \quad \mathbf{q}(x,0) = \bar{\mathbf{q}}(x) \ .$$

- Let $\mathbf{l}_i$ and $\mathbf{r}_i$ be the left and right eigenvectors of A

$$\begin{aligned} A \ \mathbf{r}_i &= v_i \mathbf{r}_i \ , \qquad & \mathbf{l}_i \ A &= v_i \mathbf{l}_i \ , \\ |\mathbf{r}_i| &= 1 \ , & \mathbf{l}_i \cdot \mathbf{r}_j &= \delta_{ij} \ . \end{aligned}$$

Appetizer: linear systems

- n conservation laws: A is a $n \times n$ matrix with real eigenvalues $v_1 < \cdots < v_n$

$$\partial_t \mathbf{q} + A \partial_x \mathbf{q} = 0 \ , \quad \mathbf{q}(x,0) = \bar{\mathbf{q}}(x) \ .$$

- Let $\mathbf{l}_i$ and $\mathbf{r}_i$ be the left and right eigenvectors of A

$$\begin{aligned} A\ \mathbf{r}_i &= v_i \mathbf{r}_i \ , \qquad & \mathbf{l}_i\ A &= v_i \mathbf{l}_i \ , \\ |\mathbf{r}_i| &= 1 \ , & \mathbf{l}_i \cdot \mathbf{r}_j &= \delta_{ij} \ . \end{aligned}$$

- Let $\lambda_i = \mathbf{l}_i \cdot \mathbf{q}$, the coordinates of $\mathbf{q}$ w.r.t the basis $\{\mathbf{r}_1, \cdots, \mathbf{r}_n\}$

$$\partial_t \lambda_i + v_i \partial_x \lambda_i = 0 \ , \quad \lambda_i(0,x) = \bar{\lambda}_i(x) = \mathbf{l}_i \cdot \bar{\mathbf{q}}(x) \ .$$

Appetizer: linear systems

- n conservation laws: A is a $n \times n$ matrix with real eigenvalues $v_1 < \cdots < v_n$

$$\partial_t \mathbf{q} + A \partial_x \mathbf{q} = 0 \ , \quad \mathbf{q}(x,0) = \bar{\mathbf{q}}(x) \ .$$

- Let $\mathbf{l}_i$ and $\mathbf{r}_i$ be the left and right eigenvectors of A

$$\begin{aligned} A\ \mathbf{r}_i &= v_i \mathbf{r}_i \ , \qquad & \mathbf{l}_i\ A &= v_i \mathbf{l}_i \ , \\ |\mathbf{r}_i| &= 1 \ , & \mathbf{l}_i \cdot \mathbf{r}_j &= \delta_{ij} \ . \end{aligned}$$

- Let $\lambda_i = \mathbf{l}_i \cdot \mathbf{q}$, the coordinates of $\mathbf{q}$ w.r.t the basis $\{\mathbf{r}_1, \cdots, \mathbf{r}_n\}$

$$\partial_t \lambda_i + v_i \partial_x \lambda_i = 0 \ , \quad \lambda_i(0,x) = \bar{\lambda}_i(x) = \mathbf{l}_i \cdot \bar{\mathbf{q}}(x) \ .$$

- Solution of the original system

$$\mathbf{q}(t,x) = \sum_{i=1}^{n} \bar{\lambda}_i(x - v_i t) \mathbf{r}_i \ .$$

The simplest nonlinear conservation law: Hopf equation

- The Hopf / inviscid Burgers equation

$$\partial_t q + q \partial_x q = 0 \ , \quad q(0,x) = \bar{q}(x) \ .$$

The simplest nonlinear conservation law: Hopf equation

- The Hopf / inviscid Burgers equation

$$\partial_t q + q\partial_x q = 0 \ , \quad q(0,x) = \bar{q}(x) \ .$$

- The field q is constant along the characteristic curve $\frac{\mathrm{d}x}{\mathrm{d}t} = q$

$$q\left(t, x + t\bar{q}(x)\right) = \bar{q}(x) \ .$$

The simplest nonlinear conservation law: Hopf equation

- The Hopf / inviscid Burgers equation

$$\partial_t q + q\partial_x q = 0 \ , \quad q(0,x) = \bar{q}(x) = \frac{1}{1+x^2} \ .$$

- The field q is constant along the characteristic curve $\frac{\mathrm{d}x}{\mathrm{d}t} = q$

$$q\left(t, x + \frac{t}{1+x^2}\right) = \frac{1}{1+x^2} \ .$$

The simplest nonlinear conservation law: Hopf equation

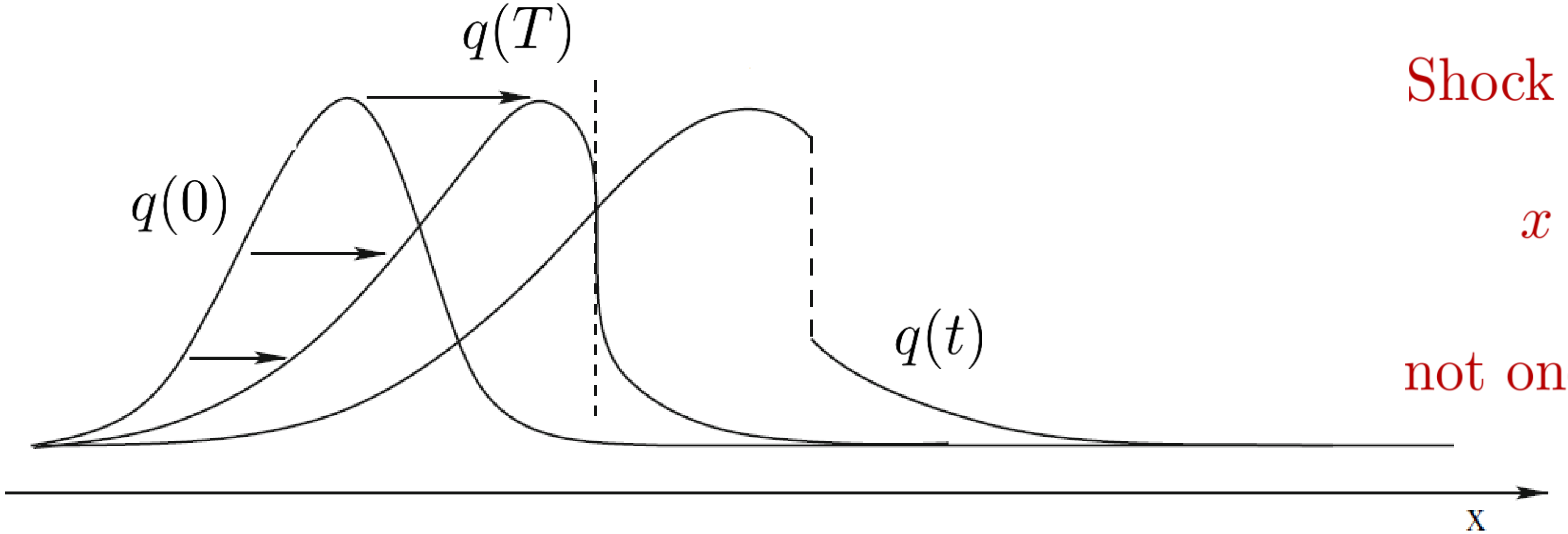
- The Hopf / inviscid Burgers equation

$$\partial_t q + q\partial_x q = 0 \ , \quad q(0,x) = \bar{q}(x) = \frac{1}{1+x^2} \ .$$

- The field q is constant along the characteristic curve $\frac{\mathrm{d}x}{\mathrm{d}t} = q$

$$q\left(t, x + \frac{t}{1+x^2}\right) = \frac{1}{1+x^2} \ .$$

[Dafermos (2005)]
[Bressan (2009)]



Shock at $t = T = 8/\sqrt{27}$

$$x \mapsto x + \frac{t}{1+x^2}$$

not one-to-one for $t > T$.

Riemann invariants

- n conservation laws: $A(\mathbf{q})$ is $n \times n$ with real eigenvalues $v_1(\mathbf{q}) < \cdots < v_n(\mathbf{q})$

$$\partial_t \mathbf{q} + A(\mathbf{q}) \partial_x \mathbf{q} = 0 \ , \quad \mathbf{q}(x, 0) = \bar{\mathbf{q}}(x) \ .$$

Riemann invariants

- n conservation laws: $A(\mathbf{q})$ is $n \times n$ with real eigenvalues $v_1(\mathbf{q}) < \cdots < v_n(\mathbf{q})$

$$\partial_t \mathbf{q} + A(\mathbf{q}) \partial_x \mathbf{q} = 0 \ , \quad \mathbf{q}(x,0) = \bar{\mathbf{q}}(x) \ .$$

- Riemann invariants λ_j are constant along the characteristics $\frac{\mathrm{d}x}{\mathrm{d}t} = v_j$

$$\partial_t \lambda_j + v_j(\lambda_0, \cdots, \lambda_n) \partial_x \lambda_j = 0.$$

Riemann invariants

- n conservation laws: $A(\mathbf{q})$ is $n \times n$ with real eigenvalues $v_1(\mathbf{q}) < \cdots < v_n(\mathbf{q})$

$$\partial_t \mathbf{q} + A(\mathbf{q}) \partial_x \mathbf{q} = 0 \ , \quad \mathbf{q}(x,0) = \bar{\mathbf{q}}(x) \ .$$

- Riemann invariants λ_j are constant along the characteristics $\frac{\mathrm{d}x}{\mathrm{d}t} = v_j$

$$\partial_t \lambda_j + v_j(\lambda_0, \cdots, \lambda_n) \partial_x \lambda_j = 0.$$

Always exist for $n = 2$!

Generically don't for $n \geq 3$...

Riemann invariants

- n conservation laws: $A(\mathbf{q})$ is $n \times n$ with real eigenvalues $v_1(\mathbf{q}) < \cdots < v_n(\mathbf{q})$

$$\partial_t \mathbf{q} + A(\mathbf{q})\partial_x \mathbf{q} = 0 \ , \quad \mathbf{q}(x,0) = \bar{\mathbf{q}}(x) \ .$$

- Riemann invariants λ_j are constant along the characteristics $\frac{\mathrm{d}x}{\mathrm{d}t} = v_j$

$$\partial_t \lambda_j + v_j(\lambda_0, \cdots, \lambda_n)\partial_x \lambda_j = 0.$$

Always exist for $n = 2$!

Generically don't for $n \geq 3$...

- Genuine nonlinearity

$$\partial_{\lambda_j} v_j \neq 0 \ .$$

VS

- Linear degeneracy

$$\partial_{\lambda_j} v_j = 0 \ .$$

Riemann invariants

- n conservation laws: $A(\mathbf{q})$ is $n \times n$ with real eigenvalues $v_1(\mathbf{q}) < \cdots < v_n(\mathbf{q})$

$$\partial_t \mathbf{q} + A(\mathbf{q})\partial_x \mathbf{q} = 0 \ , \quad \mathbf{q}(x,0) = \bar{\mathbf{q}}(x) \ .$$

- Riemann invariants λ_j are constant along the characteristics $\frac{\mathrm{d}x}{\mathrm{d}t} = v_j$

$$\partial_t \lambda_j + v_j(\lambda_0, \cdots, \lambda_n)\partial_x \lambda_j = 0.$$

Always exist for $n = 2$!

Generically don't for $n \geq 3$...

- Genuine nonlinearity

$$\partial_{\lambda_j} v_j \neq 0 \ .$$

VS

- Linear degeneracy

$$\partial_{\lambda_j} v_j = 0 \ .$$

No shocks!

[No shocks in GHD: Hübner, Doyon (2023)]

Exact solutions: the hodograph method

- Integrable via the hodograph transform if semi-Hamiltonian

$$\partial_{\lambda_j} \frac{\partial_{\lambda_k} v_i}{v_k - v_i} = \partial_{\lambda_k} \frac{\partial_{\lambda_j} v_i}{v_j - v_i} \ , \quad i \neq j \neq k \ .$$

Exact solutions: the hodograph method

- Integrable via the hodograph transform if semi-Hamiltonian

$$\partial_{\lambda_j} \frac{\partial_{\lambda_k} v_i}{v_k - v_i} = \partial_{\lambda_k} \frac{\partial_{\lambda_j} v_i}{v_j - v_i} \ , \quad i \neq j \neq k \ .$$

- Hodograph transform

$$x - v_j(\lambda_1, \cdots, \lambda_n)t = w_j(\lambda_1, \cdots, \lambda_n) \ ,$$

where the functions w_j solve the linear system

$$\frac{\partial_{\lambda_k} w_j}{w_k - w_j} = \frac{\partial_{\lambda_k} v_j}{v_k - v_j} \ , \quad j \neq k \ .$$

Example: Bose-Einstein condensate

- Large-scale dynamics of non-uniform Bose gas at zero temperature described by NLS

$$i\partial_t\psi = -\frac{1}{2}\partial_{xx}\psi + |\psi|^2\psi \ .$$

Example: Bose-Einstein condensate

- Large-scale dynamics of non-uniform Bose gas at zero temperature described by NLS

$$i\partial_t\psi = -\frac{1}{2}\partial_{xx}\psi + |\psi|^2\psi \ .$$

- Madelung transform

$$\psi(x,t) = \sqrt{\rho(x,t)}e^{iS(x,t)} \ , \quad \text{and} \quad u(x,t) = \partial_x S(x,t) \ ,$$

$$\Rightarrow \quad \begin{cases} \partial_t\rho + \partial_x(\rho u) = 0 \\ \partial_t u + \partial_x\left[\dfrac{u^2}{2} + \underbrace{\rho}_{\text{Effective pressure}} - \underbrace{\dfrac{1}{\sqrt{\rho}}\partial_{xx}\sqrt{\rho}}_{\text{Quantum pressure}}\right] = 0 \end{cases} .$$

Example: Bose-Einstein condensate

- Large-scale dynamics of non-uniform Bose gas at zero temperature described by NLS

$$i\partial_t\psi = -\frac{1}{2}\partial_{xx}\psi + |\psi|^2\psi \ .$$

- Madelung transform

$$\psi(x,t) = \sqrt{\rho(x,t)}e^{iS(x,t)} \ , \quad \text{and} \quad u(x,t) = \partial_x S(x,t) \ ,$$

$$\Rightarrow \quad \begin{cases} \partial_t\rho + \partial_x(\rho u) = 0 \\ \partial_t u + \partial_x\left[\frac{u^2}{2} + \underbrace{\rho}_{\text{Effective pressure}} - \underbrace{\cancel{\frac{1}{\sqrt{\rho}}\partial_{xx}\sqrt{\rho}}}_{\text{Quantum pressure}}\right] = 0 \end{cases} .$$

Example: Bose-Einstein condensate

- Large-scale dynamics of non-uniform Bose gas at zero temperature described by NLS

$$i\partial_t\psi = -\frac{1}{2}\partial_{xx}\psi + |\psi|^2\psi \ .$$

- Madelung transform

$$\psi(x,t) = \sqrt{\rho(x,t)}e^{iS(x,t)} \ , \quad \text{and} \quad u(x,t) = \partial_x S(x,t) \ ,$$

$$\Rightarrow \quad \begin{cases} \partial_t\rho + \partial_x(\rho u) = 0 \\ \partial_t u + \partial_x\left[\dfrac{u^2}{2} + \rho - \xcancel{\dfrac{1}{\sqrt{\rho}}\partial_{xx}}\sqrt{\rho}\right] = 0 \end{cases} .$$

- Remark: for polytropic gases replace the pressure by $\rho^{(\gamma-1)/2}$.

Example: Bose-Einstein condensate

1. Quasilinear form

$$\partial_t \begin{pmatrix} \rho \\ u \end{pmatrix} + \begin{pmatrix} u & \rho \\ 1 & u \end{pmatrix} \partial_x \begin{pmatrix} \rho \\ u \end{pmatrix} = 0 \ .$$

Example: Bose-Einstein condensate

1. Quasilinear form

$$\partial_t \begin{pmatrix} \rho \\ u \end{pmatrix} + \begin{pmatrix} u & \rho \\ 1 & u \end{pmatrix} \partial_x \begin{pmatrix} \rho \\ u \end{pmatrix} = 0 \ .$$

2. Characteristic velocities

$$\det \begin{pmatrix} u - v_\pm & \rho \\ 1 & u - v_\pm \end{pmatrix} = 0 \quad \Rightarrow \quad v_\pm = u \pm \sqrt{\rho} \ .$$

Example: Bose-Einstein condensate

1. Quasilinear form

$$\partial_t \begin{pmatrix} \rho \\ u \end{pmatrix} + \begin{pmatrix} u & \rho \\ 1 & u \end{pmatrix} \partial_x \begin{pmatrix} \rho \\ u \end{pmatrix} = 0 \ .$$

2. Characteristic velocities

$$\det \begin{pmatrix} u - v_\pm & \rho \\ 1 & u - v_\pm \end{pmatrix} = 0 \quad \Rightarrow \quad v_\pm = u \pm \sqrt{\rho} \ .$$

3. Riemann invariants

$$\begin{cases} \lambda_\pm = u \pm 2\sqrt{\rho} \\ \partial_t \lambda_+ + \left(\frac{3}{4}\lambda_+ + \frac{1}{4}\lambda_- \right) \partial_x \lambda_+ = 0 \\ \partial_t \lambda_- + \left(\frac{3}{4}\lambda_- + \frac{1}{4}\lambda_+ \right) \partial_x \lambda_- = 0 \end{cases} \ .$$

Example: Bose-Einstein condensate

4. Hodograph transform: $x - v_\pm t = w_\pm$

$$\begin{cases} \dfrac{\partial_{\lambda_-} w_+}{w_+ - w_-} = \dfrac{\partial_{\lambda_-} v_+}{v_+ - v_-} \\ \dfrac{\partial_{\lambda_+} w_-}{w_- - w_+} = \dfrac{\partial_{\lambda_+} v_-}{v_- - v_+} \end{cases} .$$

Example: Bose-Einstein condensate

4. Hodograph transform: $x - v_\pm t = w_\pm$

$$\begin{cases} \dfrac{\partial_{\lambda_-} w_+}{w_+ - w_-} = \dfrac{\partial_{\lambda_-} v_+}{v_+ - v_-} = \dfrac{1}{2(\lambda_+ - \lambda_-)} \\ \dfrac{\partial_{\lambda_+} w_-}{w_+ - w_-} = \dfrac{\partial_{\lambda_+} v_-}{v_+ - v_-} = \dfrac{1}{2(\lambda_+ - \lambda_-)} \end{cases} \Rightarrow \quad \partial_{\lambda_+} w_- = \partial_{\lambda_-} w_+ \ .$$

Example: Bose-Einstein condensate

4. Hodograph transform: $x - v_\pm t = w_\pm$

$$\begin{cases} \dfrac{\partial_{\lambda_-} w_+}{w_+ - w_-} = \dfrac{\partial_{\lambda_-} v_+}{v_+ - v_-} = \dfrac{1}{2(\lambda_+ - \lambda_-)} \\ \dfrac{\partial_{\lambda_+} w_-}{w_+ - w_-} = \dfrac{\partial_{\lambda_+} v_-}{v_+ - v_-} = \dfrac{1}{2(\lambda_+ - \lambda_-)} \end{cases} \quad \Rightarrow \quad \partial_{\lambda_+} w_- = \partial_{\lambda_-} w_+ \ .$$

5. Potential representation: $w_\pm = \partial_{\lambda_\pm} W$

$$\frac{\partial^2 W}{\partial\lambda_+ \partial\lambda_-} - \frac{1}{2(\lambda_+ - \lambda_-)} \left(\frac{\partial W}{\partial\lambda_+} - \frac{\partial W}{\partial\lambda_-} \right) = 0 \ .$$

Euler-Poisson-Darboux equation
(solvable via Riemann-Volterra method)

[Sommerfeld (1964)]

Example: Bose-Einstein condensate

4. Hodograph transform: $x - v_{\pm}t = w_{\pm}$

$$\begin{cases} \dfrac{\partial_{\lambda_-} w_+}{w_+ - w_-} = \dfrac{\partial_{\lambda_-} v_+}{v_+ - v_-} = \dfrac{1}{2(\lambda_+ - \lambda_-)} \\ \dfrac{\partial_{\lambda_+} w_-}{w_+ - w_-} = \dfrac{\partial_{\lambda_+} v_-}{v_+ - v_-} = \dfrac{1}{2(\lambda_+ - \lambda_-)} \end{cases} \quad \Rightarrow \quad \partial_{\lambda_+} w_- = \partial_{\lambda_-} w_+ \ .$$

5. Potential representation: $w_{\pm} = \partial_{\lambda_{\pm}} W$

Euler-Poisson-Darboux equation

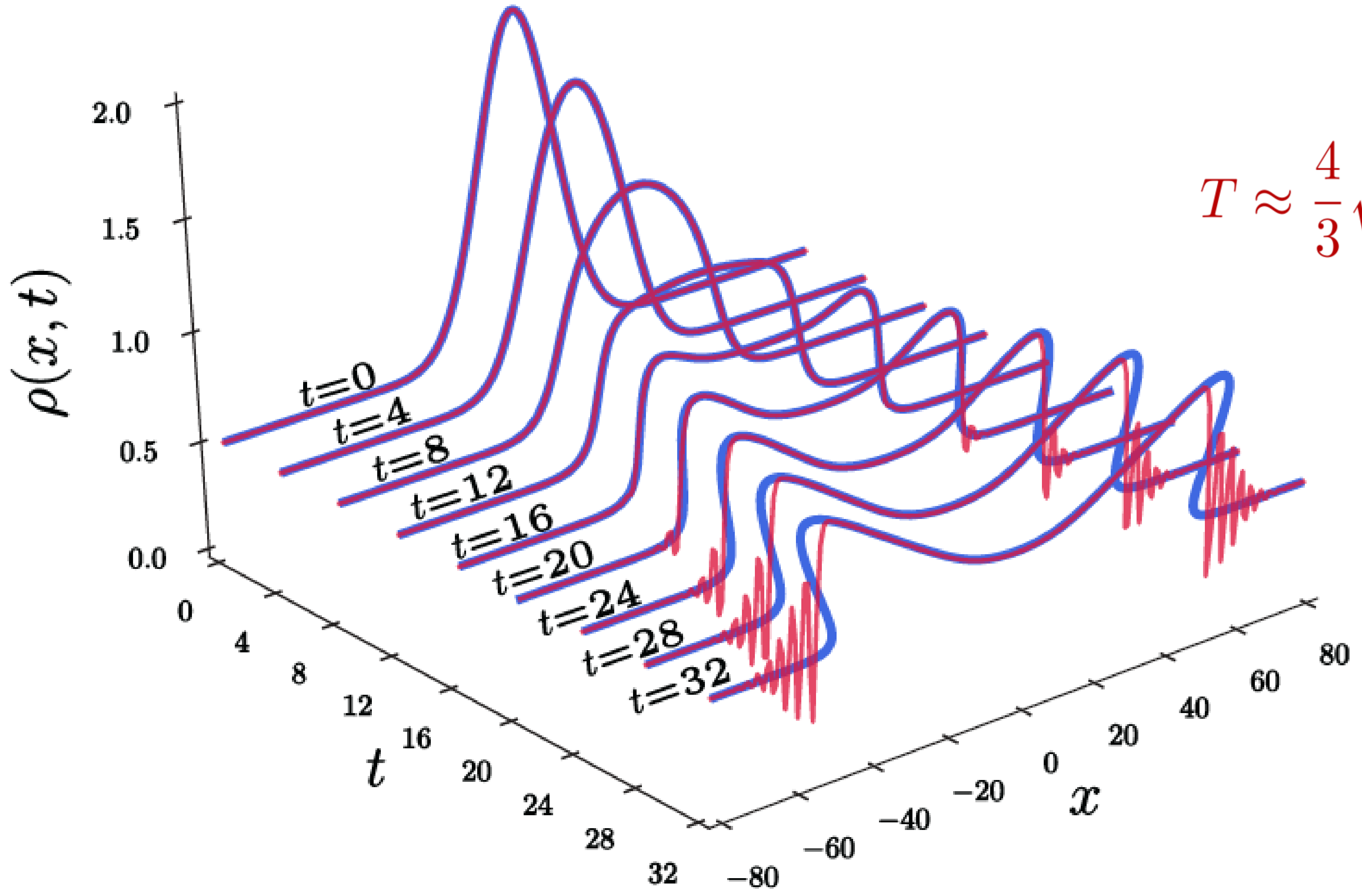
$$\frac{\partial^2 W}{\partial\lambda_+ \partial\lambda_-} - \frac{1}{2(\lambda_+ - \lambda_-)} \left(\frac{\partial W}{\partial\lambda_+} - \frac{\partial W}{\partial\lambda_-} \right) = 0 \ .$$

6. Invert the hodograph transform

$$\lambda_{\pm} = u \pm 2\sqrt{\rho} \quad \Rightarrow \quad \rho = \frac{(\lambda_+ - \lambda_-)^2}{16} = \frac{1}{32} \left(\frac{\partial_{\lambda_+} W - \partial_{\lambda_-} W}{\partial_{\lambda_+ \lambda_-} W} \right)^2 \ .$$

Example: Bose-Einstein condensate

7. Results for initial condition: $\bar{\rho}(x) = \rho_0 + \rho_1 \exp(-x^2/x_0^2)$



$$T \approx \frac{4}{3}\sqrt{\frac{e}{2}}x_0 \frac{\sqrt{\rho_0 + \rho_1 e^{-1/2}}}{\rho_1}$$

[Isoard (2020)]

Some references on hyperbolic conservation laws

Extensive review

- C.M. Dafermos, Hyperbolic conservation laws in continuum physics, Springer (2005).

Pedagogical introduction with a focus on shocks

- A. Bressan, Hyperbolic Conservation Laws: An Illustrated Tutorial, Springer (2009).

Modulations of wave equations

- G.B. Whitham, Linear and Nonlinear Waves. Wiley, New York (1974).
- A. Kamchatnov, Nonlinear periodic waves and their modulations: an introductory course, World Scientific (2000).

Riemann invariants and hodograph transform

- E.V. Ferapontov, Phys. Lett. A 158, 112-118, (1991).
- S.P. Tsarev, Math. USSR Izv. 37, 397–419 (1991).

Dispersive shock waves

- G.A. El, M.A. Hoefer, Physica D 333, 11-65 (2016).

Hodograph transform in elliptic systems

- T. Bonnemain, T. Gobron, D. Ullmo, Phys. Lett. A 384(25), 126608 (2020).

No shocks in GHD

- F. Hübner, B. Doyon, To be published (2023).