18TH SOUTH EAST MATHEMATICAL PHYSICS SEMINAR

KING'S College LONDON

Quadratic Mean Field Games: Schrödinger and electrostatic representations

City, University of London

Thibault Bonnemain, 16th November 2022

[Based on works w/ D. Ullmo, T. Gobron, M. Butano, C. Appert-Rolland, I. Echeverria-Huarte A. Nicolas, A. Seguin]

Game theory

Def: Mathematical framework to study strategic optimization.

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Hawk-Dove paradigm

	H	D
H	(¼ ; ¼)	(1 ; 0)
D	(0 ; 1)	(½ ; ½)

Mean Field Games

- Subdiscipline of Game Theory: problems of optimization with interacting agents in the large N limit.

- Relatively recent: seminal papers published in 2006.

[Lasry, Lions (2006)] *[Huang, Malhamé, Caines (2006)]*

- Wide litterature: mathematics, engineering sciences, economics, sociology ...

[Lachapelle, Wolfram (2011)] *[Guéant et al. (2012)]*

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⇒ Physicist approach: focus on models that can be understood completely

MFG paradigm: population dynamics

[Guéant, Lasry, Lions (2011)]

- N agents $i = 1, 2, \cdots, N \quad (N \gg 1)$

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- agent's dynamic

$$\mathrm{d}\mathbf{X}_t^i = \mathbf{a}_t^i dt + \sigma d\mathbf{w}_t^i$$

$d\mathbf{w}_t^i \equiv$ white noise
drift $\mathbf{a}_t^i \equiv$ control parameter

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$$\mathbb{E}\Big[\int_t^T \underbrace{\Big(\frac{\mu}{2}(\mathbf{a}_s^i)^2 - V[m](\mathbf{X}_s^i, \mathbf{s})\Big)}_{\text{running cost}} \mathbf{ds} + \underbrace{c_T(\mathbf{X}_T^i)}_{\text{terminal cost}}\Big]$$

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running cost — terminal cost

Mean Field Games equations

- Optimization: linear programming leads to a *(backward)* Hamilton-Jacobi-Bellman equation for the value function $u(\mathbf{x}, t)$

$$\begin{cases} \partial_t u + \frac{1}{2\mu}(\nabla_{\mathbf{x}} u)^2 + \frac{\sigma^2}{2}\Delta_{\mathbf{x}} u = V[m](x,t) \\ u(x, t{=}T) = c_T(x) \end{cases} \quad \text{(HJB)} .$$

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- Langevin dynamic $d\mathbf{X}_t^i = \mathbf{a}_t^i dt + \sigma d\mathbf{w}_t^i$ leads to a *(forward)* diffusion equation for the density $m(x,t)$

$$\begin{cases} \partial_t m - \nabla_{\mathbf{x}}\left[m \frac{\nabla_{\mathbf{x}} u}{\mu}\right] - \frac{\sigma^2}{2}\Delta_{\mathbf{x}} m = 0 \\ m(x,t{=}0) = m_0(x) \end{cases} \qquad \text{(Kolmogorov)} \; .$$

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optimal control $\mathbf{a}$

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Mean Field Game = coupling between a (collective) stochastic motion and an (individual) optimization problem through a mean field

Ergodic state

Theorem: *[Cardaliaguet, Lasry, Lions, Poretta (2013)]*

- V[m](x,t) has no explicit time dependence
- Long optimization time: $T \to \infty$
- System is confined
- ... + other conditions ...

$\Rightarrow$ for $0 \ll t \ll T$ $\quad \left| \begin{array}{l} m(\mathbf{x},t) \simeq m_{\mathrm{e}}(\mathbf{x}) \\ u(\mathbf{x},t) \simeq u_{\mathrm{e}}(\mathbf{x}) + \lambda t \end{array} \right.$

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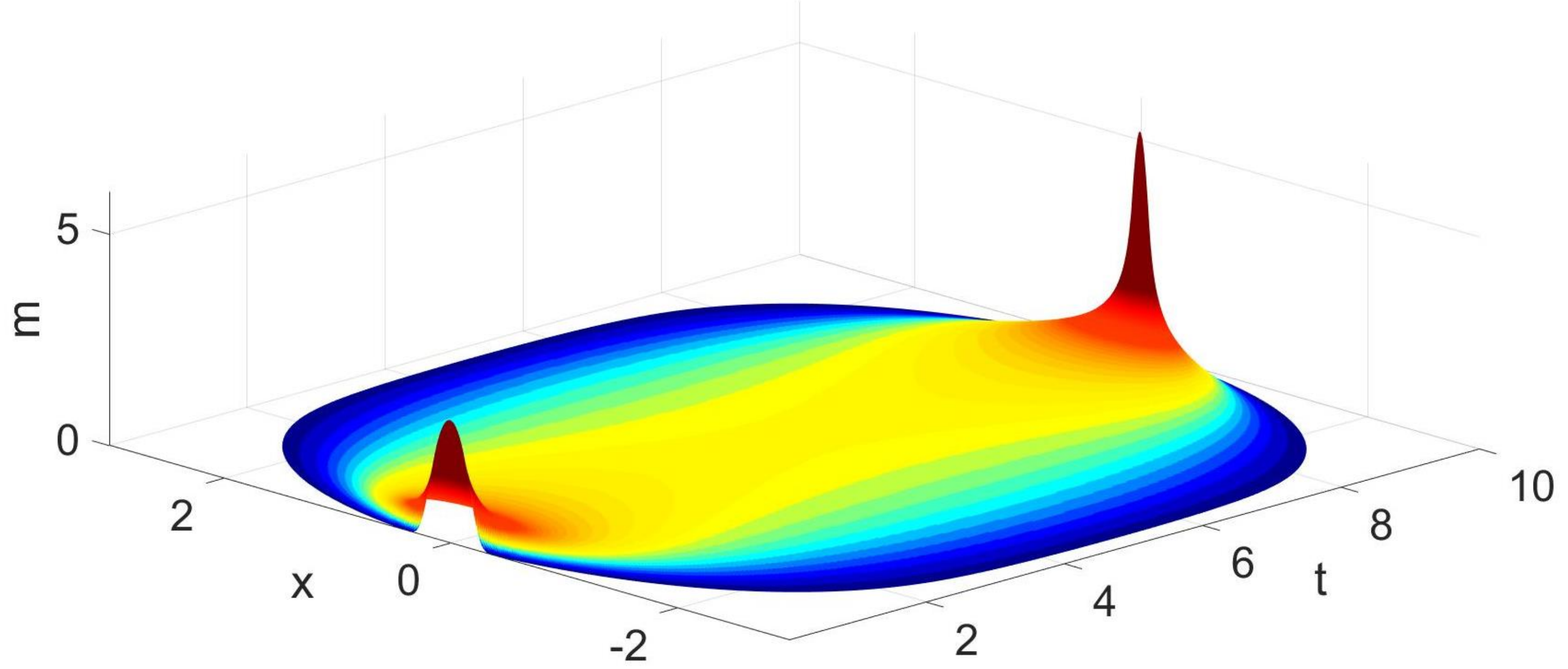
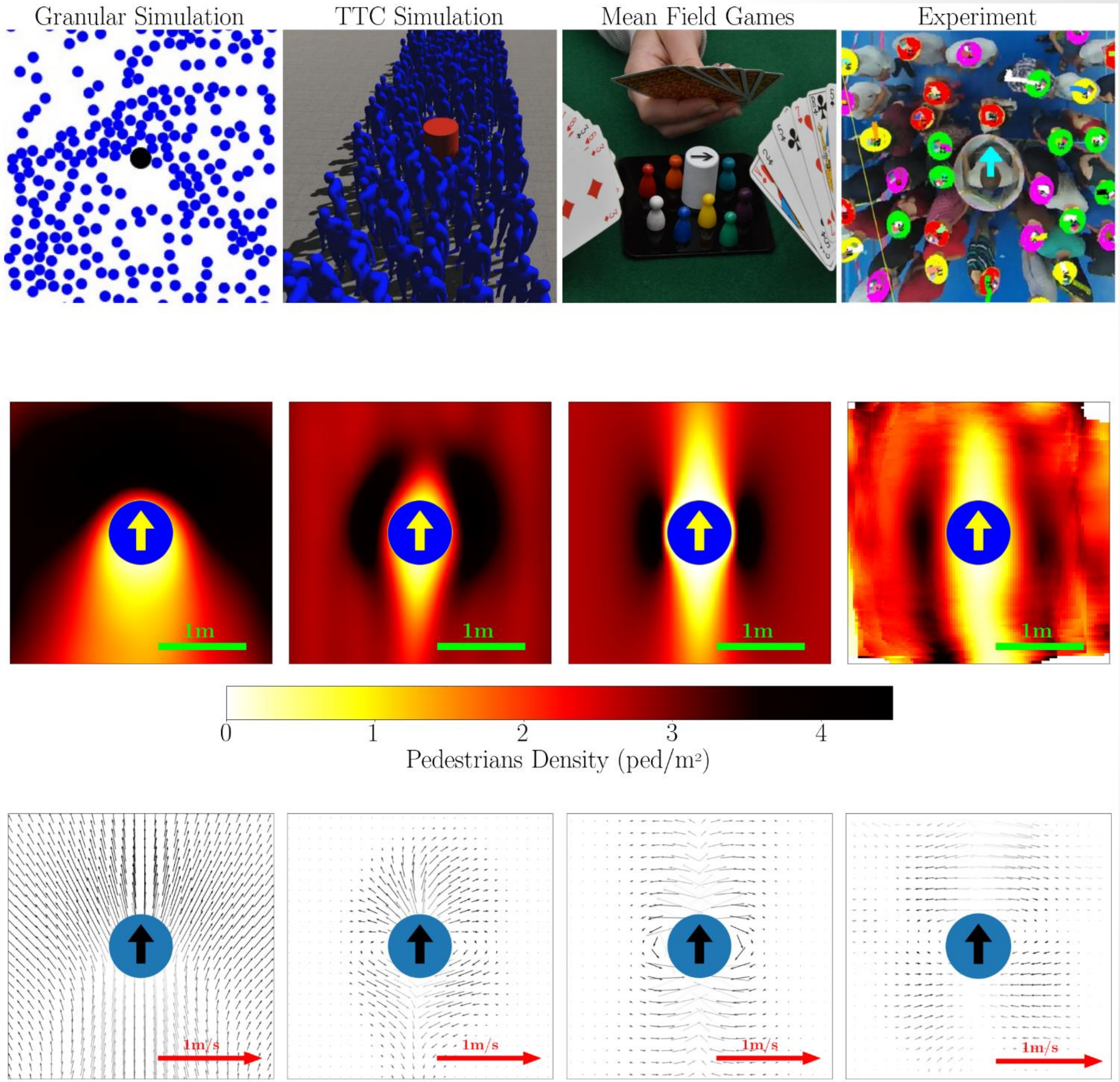
$(m_{\mathrm{e}}, u_{\mathrm{e}}, \lambda)$ such that $\begin{cases} \lambda - \dfrac{1}{2\mu}(\nabla_{\mathbf{x}} u_{\mathrm{e}})^2 + \dfrac{\sigma^2}{2}\Delta_{\mathbf{x}} u_{\mathrm{e}} = V[m_{\mathrm{e}}](\mathbf{x}) \\ \nabla_{\mathbf{x}}(m_{\mathrm{e}}(\nabla_{\mathbf{x}} u_e)) - \dfrac{\sigma^2}{2}\Delta_{\mathbf{x}} m_{\mathrm{e}} = 0 \end{cases}$

A simple example



“Search party” toy model: $V[m](x,t) = \underbrace{g}_{<0} m(x,t) + \underbrace{U_0(x)}_{\propto -x^2}$

Applications to Crowd dynamics



[Bonnemain et al. arXiv:2201.08592]

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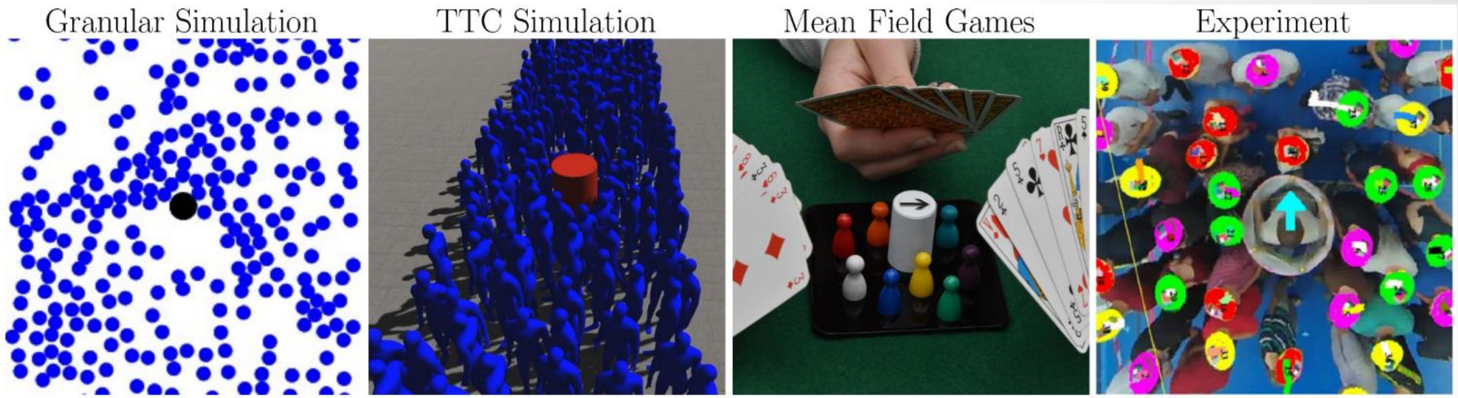
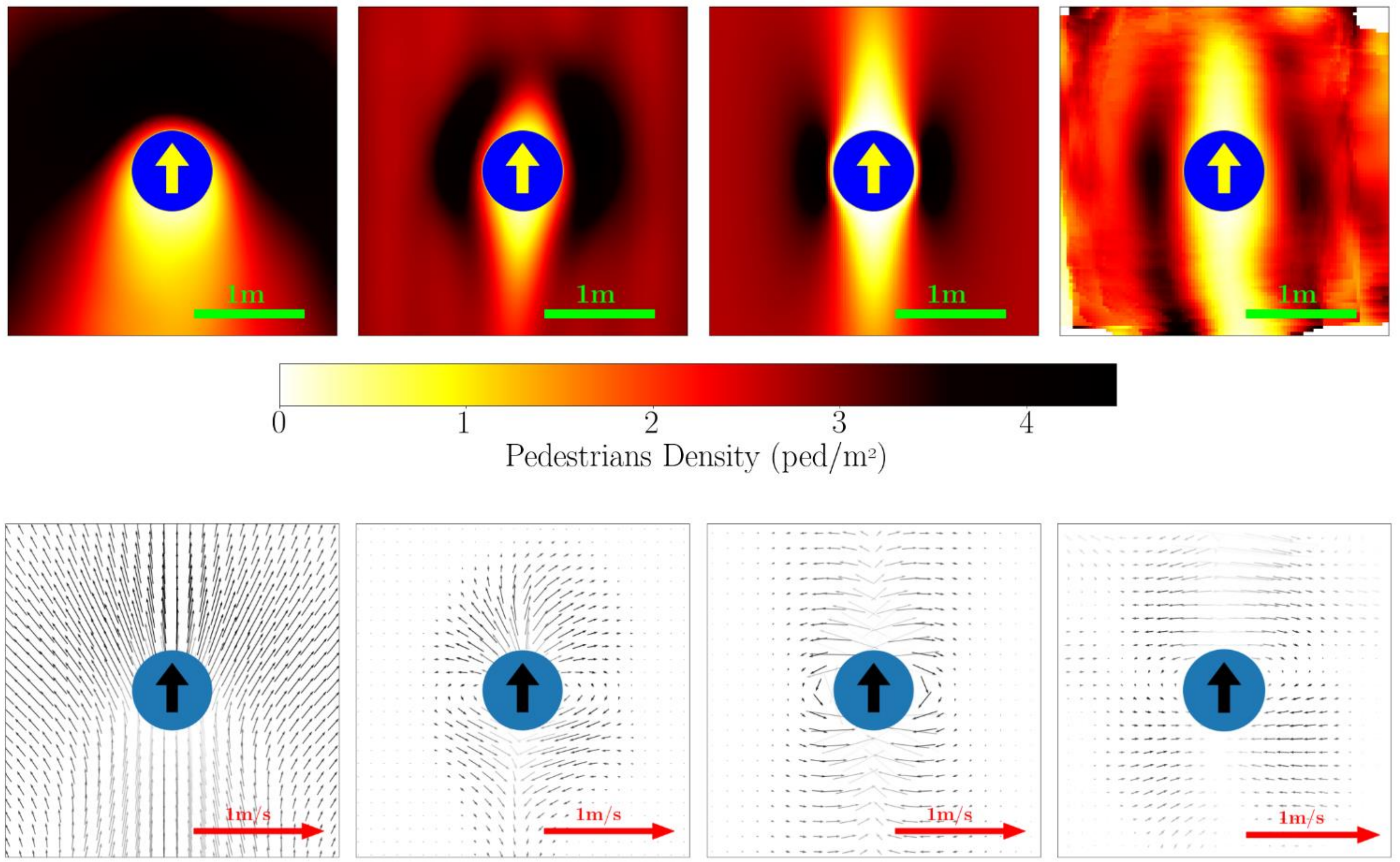


Symmetric density



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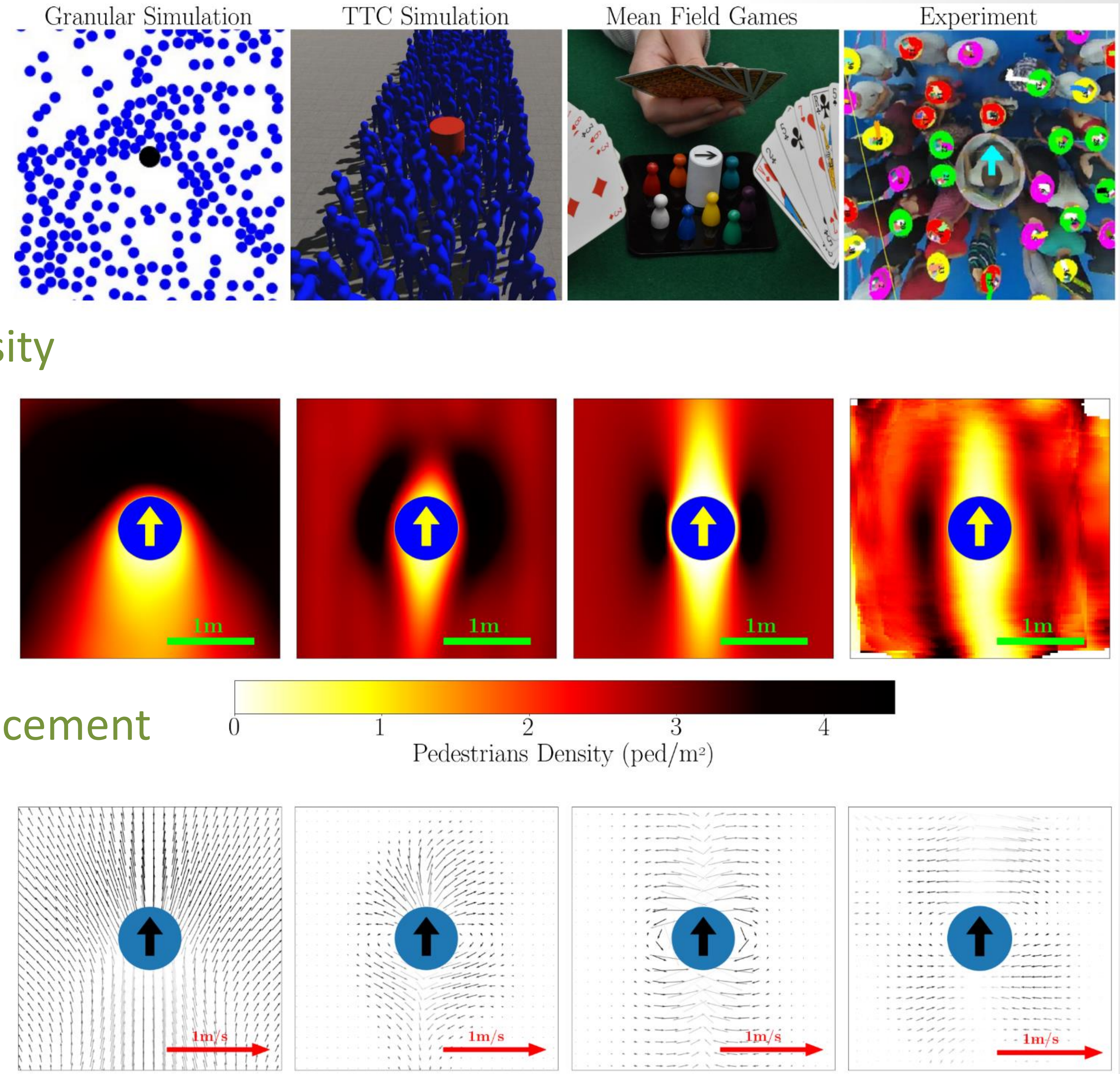
Applications to Crowd dynamics

Symmetric density

Transverse displacement



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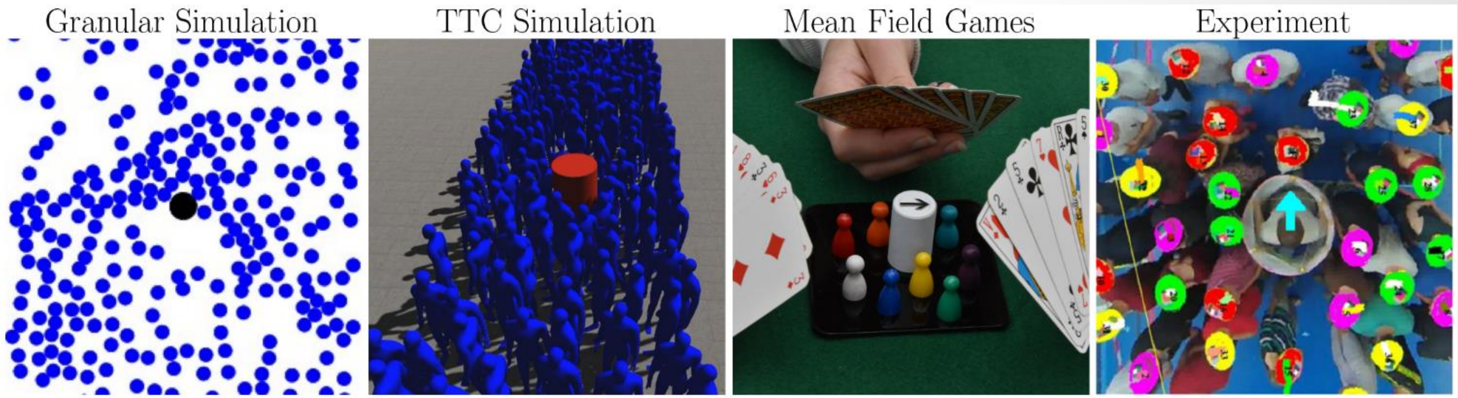
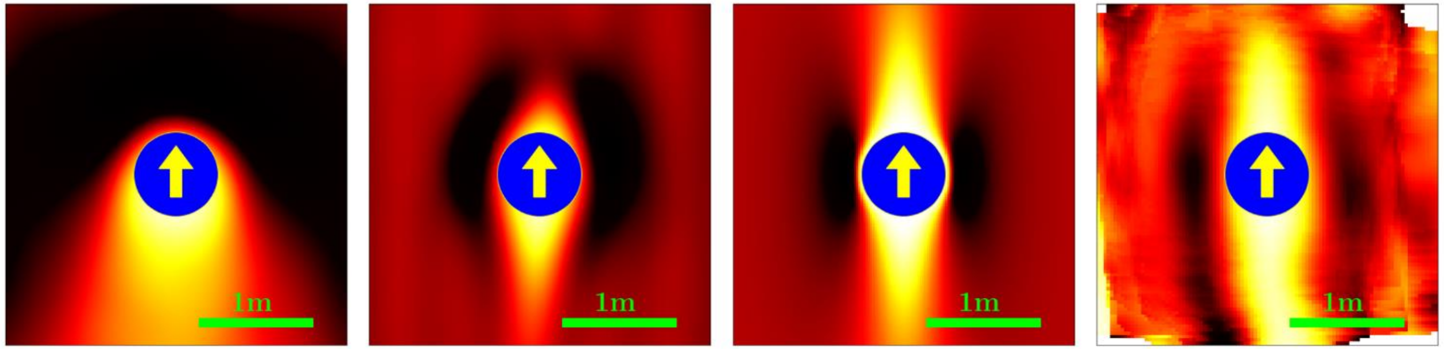
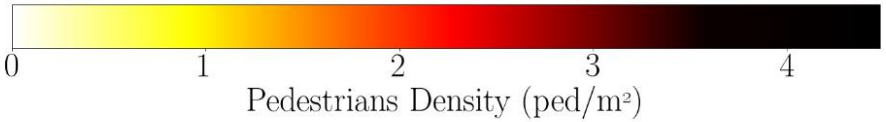


Symmetric density

Only reproduced by MFG

$$[V[m](\mathbf{x},t) = gm(\mathbf{x},t) + U_0(\mathbf{x}-\mathbf{v}t)]$$





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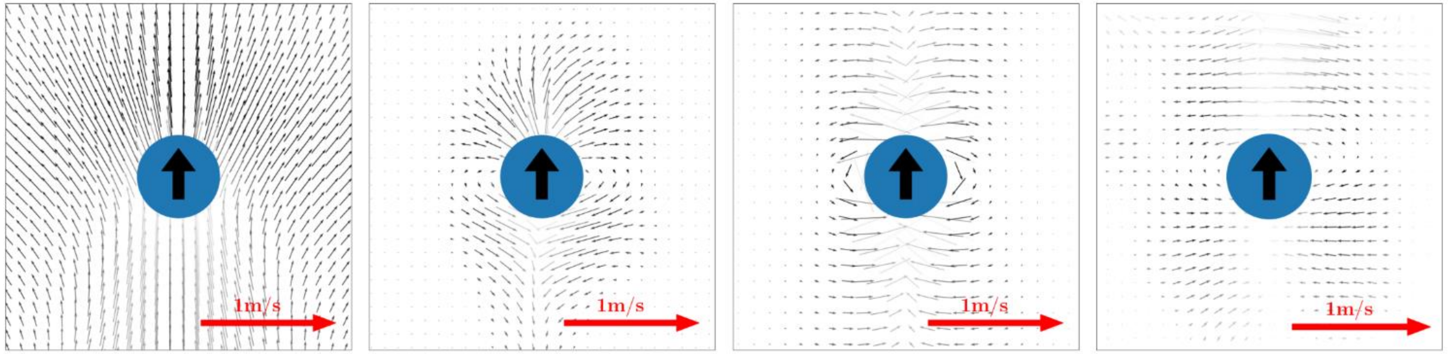

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Qualitative examples

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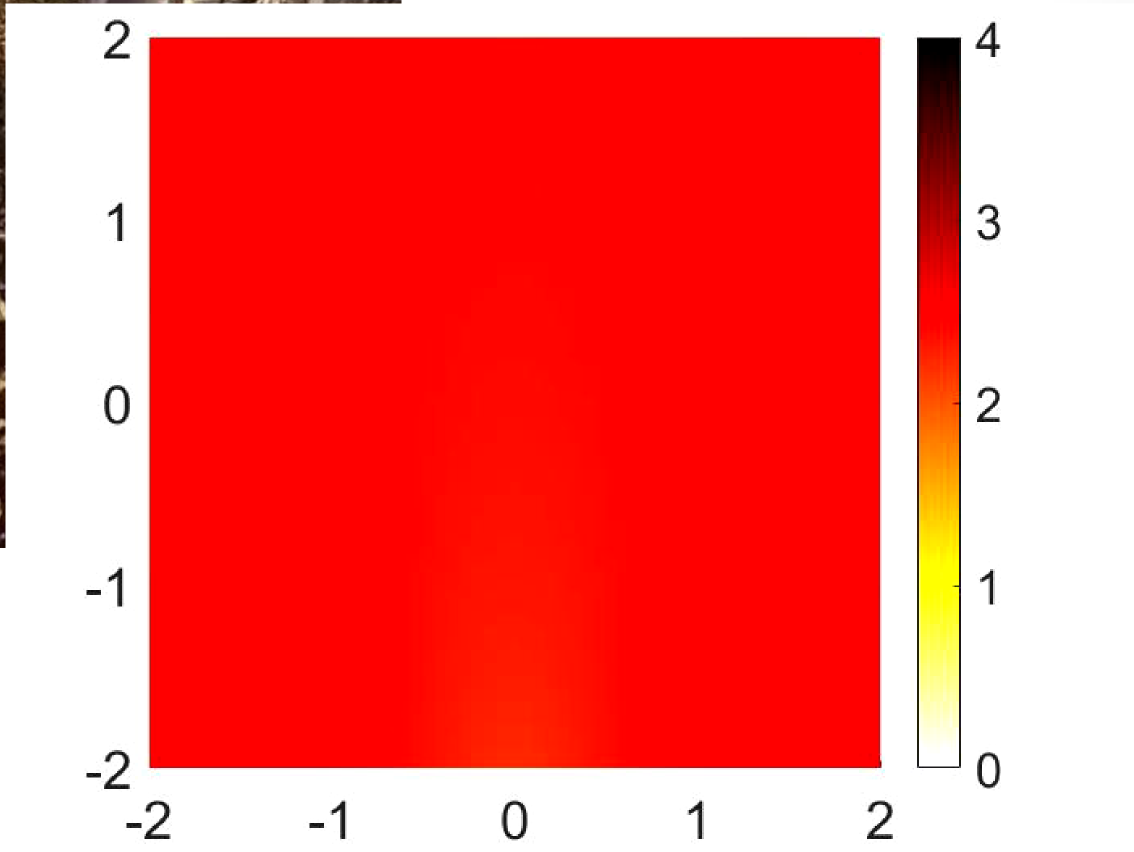

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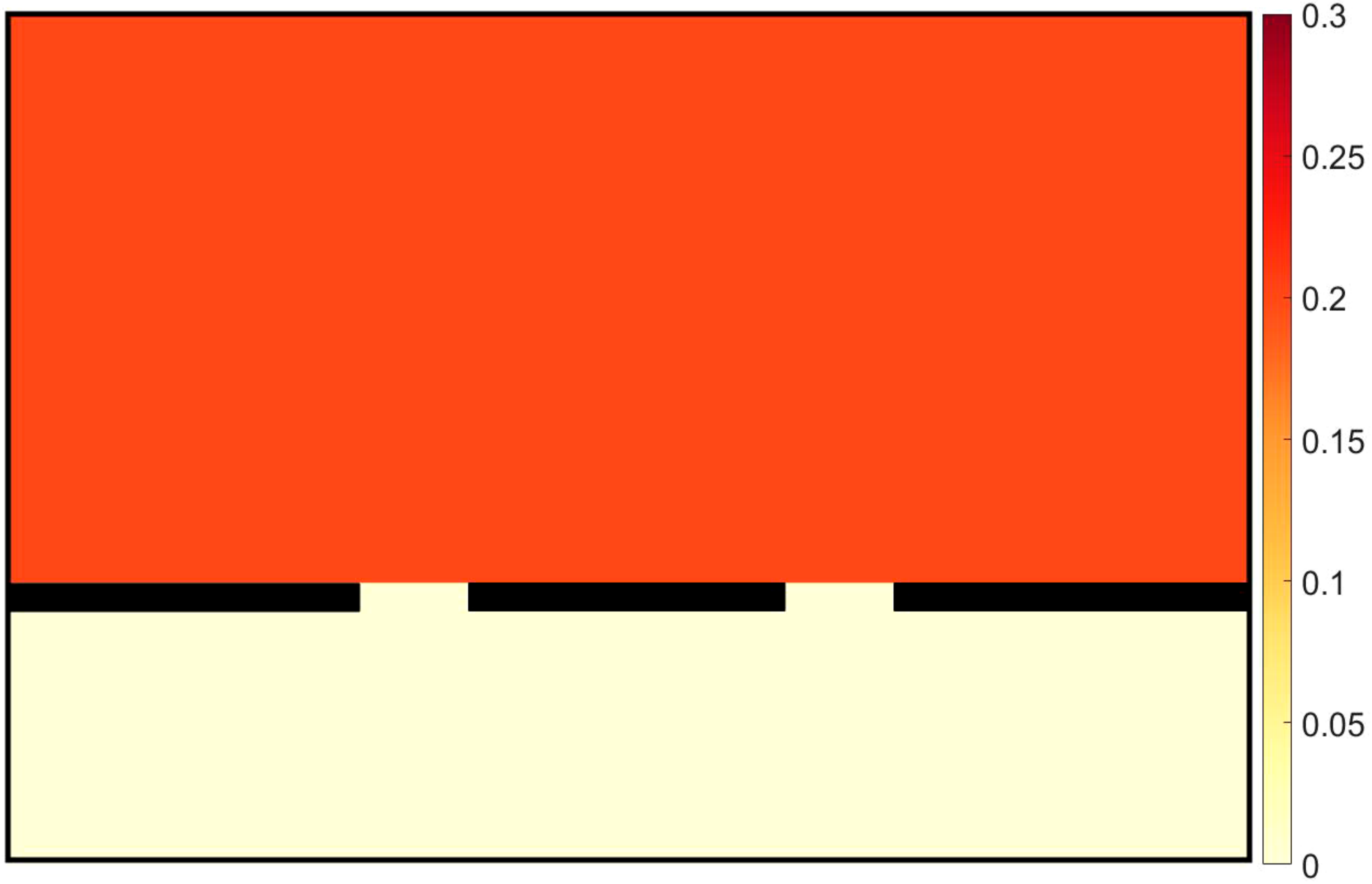

Non-linear Schrödinger representation

[Bonnemain, Gobron, Ullmo Phys.Lett. A (2020) ; SciPost (2020) ; J. Math. Phys. (2021)]

Introduce two new variables $\Phi(x,t)$, $\Gamma(x,t)$ defined by:

$$u(x,t) = -\mu\sigma^2 \log\left(\Phi(x,t)\right) \qquad m(x,t) = \Gamma(x,t)\Phi(x,t)$$

$$\Rightarrow \begin{cases} \mu\sigma^2 \partial_t \Gamma = \frac{\mu\sigma^4}{2}\Delta_{\mathbf{x}}\Gamma + U_0(\mathbf{x})\Gamma + g\, m\Gamma \\ -\mu\sigma^2 \partial_t \Phi = \frac{\mu\sigma^4}{2}\Delta_{\mathbf{x}}\Phi + U_0(\mathbf{x})\Phi + g\, m\Phi \end{cases}$$

$m = \Gamma\Phi$

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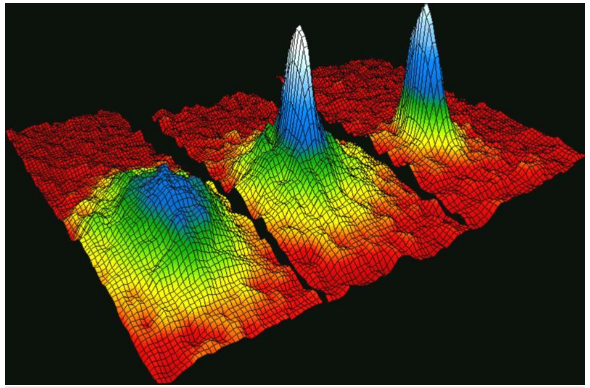

Rubidium atoms (170 nK)

$$i\hbar\partial_t\Psi = -\frac{\hbar^2}{2\mu}\Delta_{\mathbf{x}}\Psi + U_0(\mathbf{x})\Psi + g|\Psi|^2\Psi$$

Non-Linear Schrödinger

$$(\Psi, \Psi^*, \hbar) \to (\Phi, \Gamma, i\mu\sigma^2)$$

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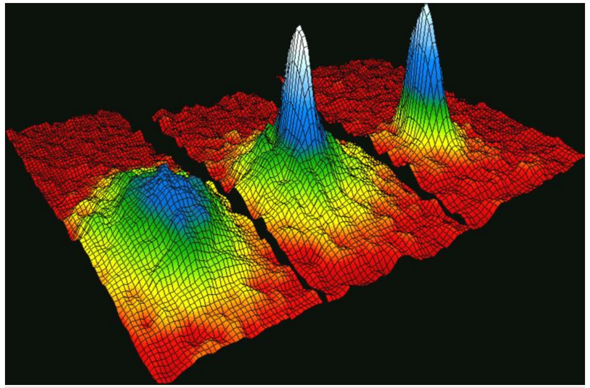

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Energy Conservation

- Quadratic MFG are variational systems

$$S[\Gamma, \Phi] \equiv \int_0^T \mathrm{d}t \int_{\mathbb{R}} \mathrm{d}x \left[\frac{\mu\sigma^2}{2} (\Gamma \partial_t \Phi - \Phi \partial_t \Gamma) \right. \\ \left. - \frac{\mu\sigma^4}{2} \nabla\Gamma . \nabla\Phi + \left[U_0 + \frac{g}{2} \Gamma\Phi \right] \Gamma\Phi \right]$$

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- Noether theorem: Energy is conserved

$$E = \int_{\mathbb{R}} \mathrm{d}x \left[-\frac{\mu\sigma^4}{2}\nabla\Gamma.\nabla\Phi + \frac{g}{2}(\Gamma\Phi)^2 + U_0\Gamma\Phi\right]$$

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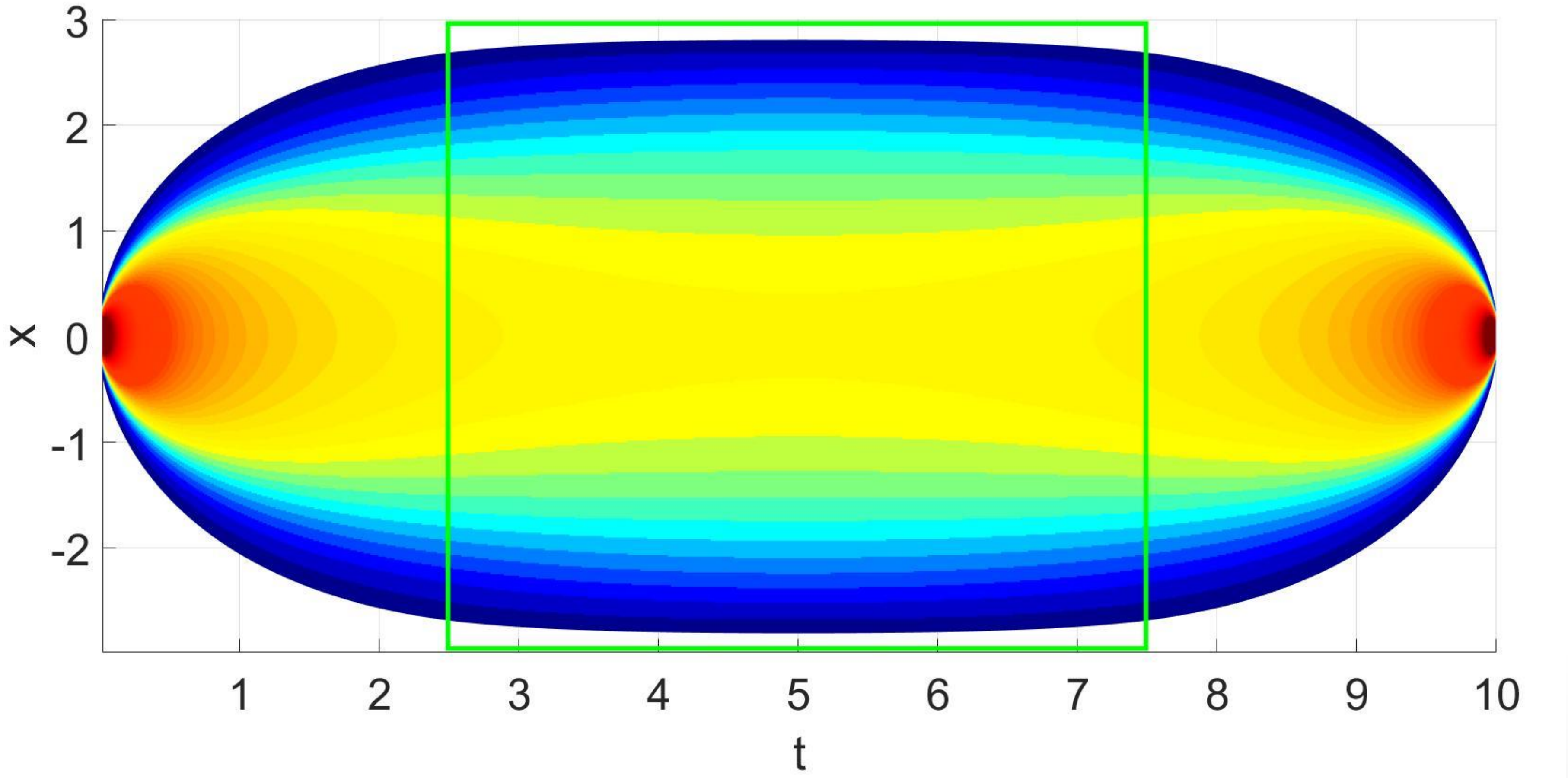
$$\Rightarrow \quad \frac{E^{\mathrm{kin}}}{E^{\mathrm{int}}} \sim \frac{\nu}{\Sigma} \qquad \nu = \frac{\mu\sigma^4}{|g|}$$

The Ergodic State



Thomas-Fermi approximation

Ergodic MFG: Schrödinger representation

$$\lambda \Psi_{\mathrm{e}} = \frac{\mu\sigma^4}{2}\Delta_x \Psi_{\mathrm{e}} + U_0(x)\Psi_{\mathrm{e}} + g\,|\Psi_{\mathrm{e}}|^2\Psi_{\mathrm{e}}$$

Thomas-Fermi approximation

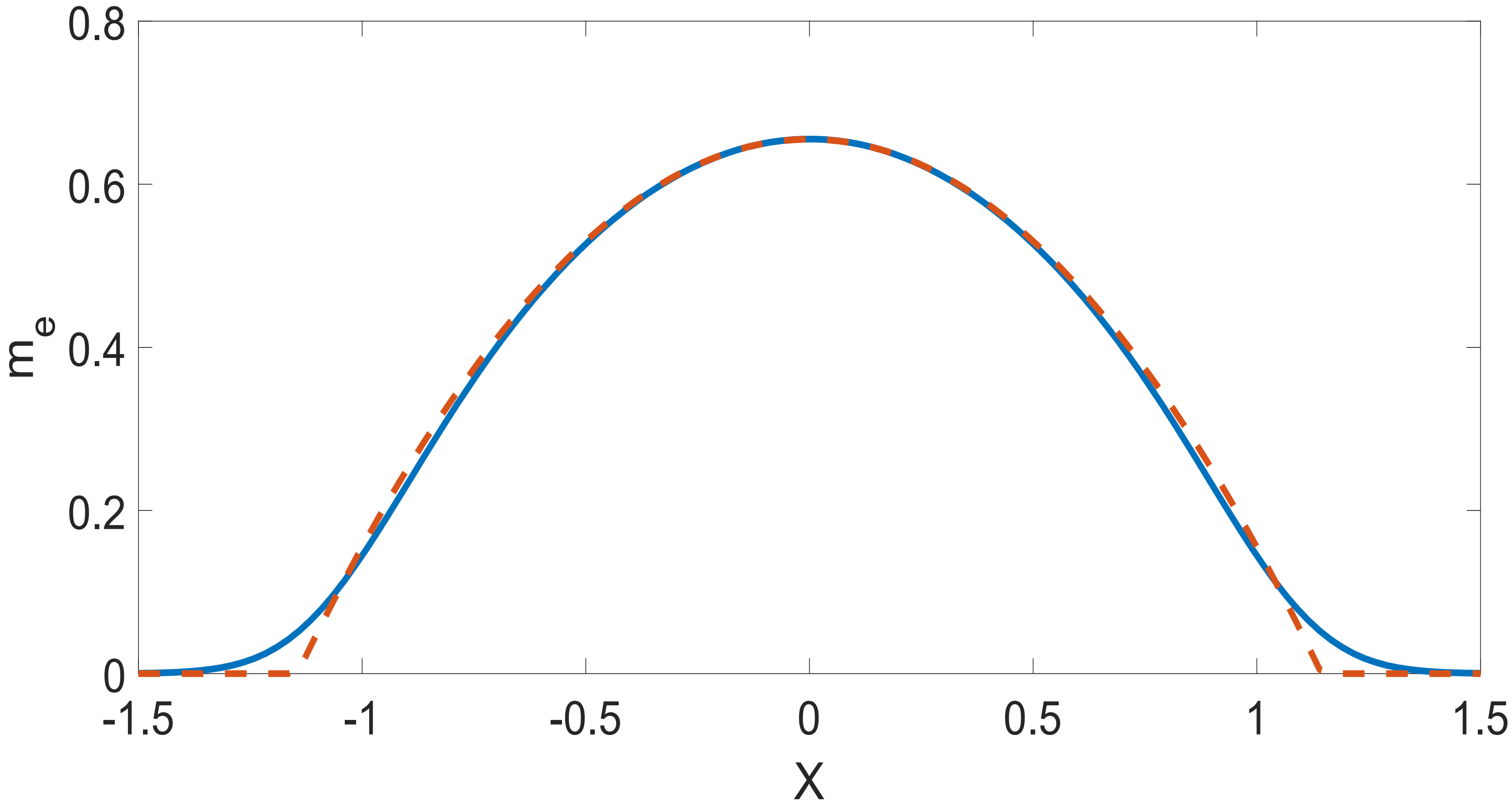
Limiting case: kinetic energy is negligible in front of interactions

$$\lambda \Psi_e = \cancel{\frac{\mu\sigma^4}{2}\Delta_x}\Psi_e + U_0(x)\Psi_e + g|\Psi_e|^2\Psi_e$$

$$\Rightarrow \quad \Psi_e = \begin{cases} \left(\frac{\lambda - U_0(x)}{|g|}\right)^{1/2} \text{if } \lambda > U_0(x) \\ 0, \text{ otherwise} \end{cases}$$

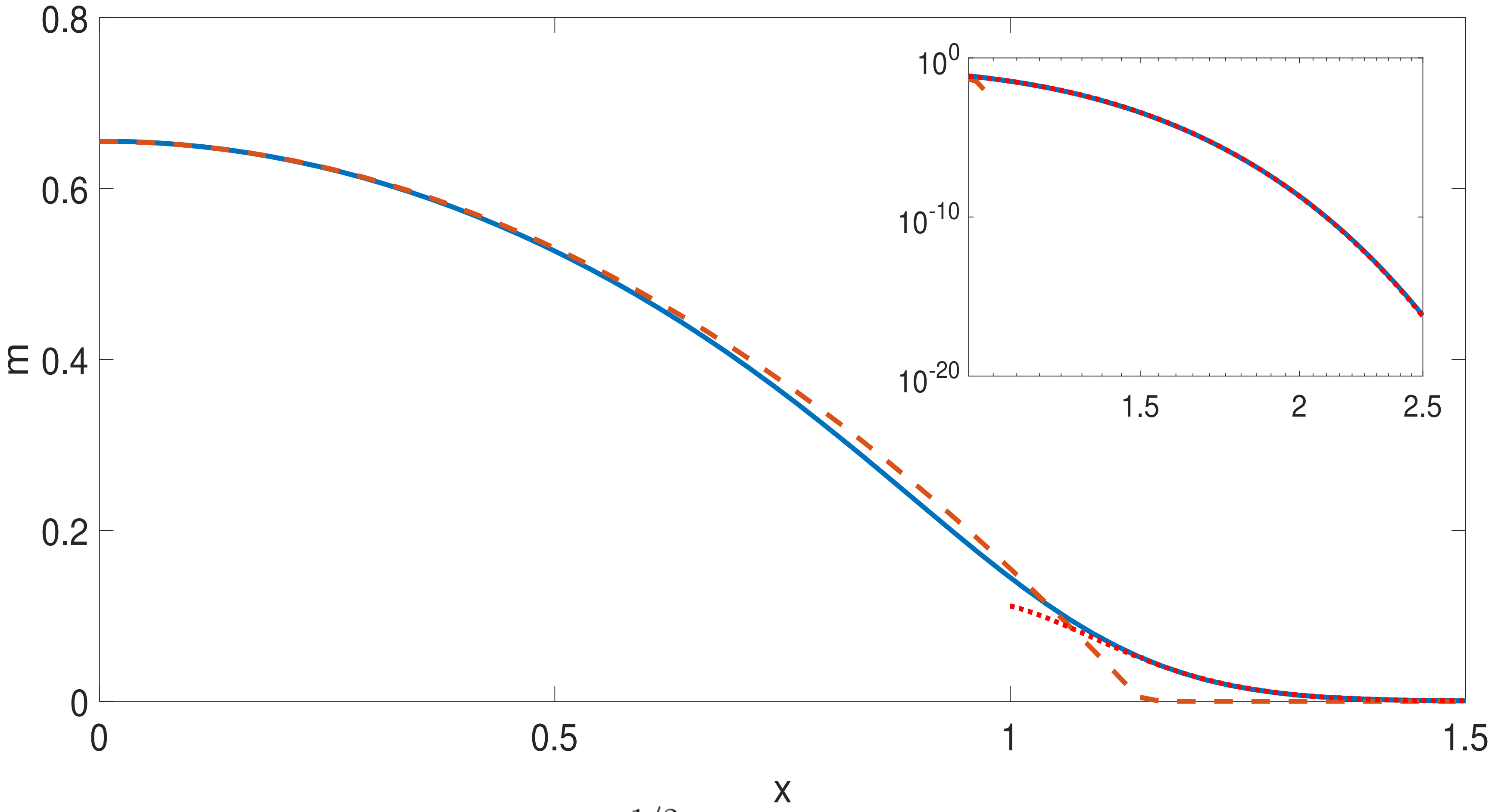
- Ψ_e represents either Φ_e or Γ_e
- $m_e = \Psi_e^2$
- λ is computed through the normalization of m_e

Thomas-Fermi approximation



$$m_{\mathrm{e}} = \begin{cases} \left(\dfrac{\lambda - U_0(x)}{|g|}\right) & \text{if } \lambda > U_0(x) \\ 0, \text{ otherwise} \end{cases} \qquad \lambda = \left(\frac{3|g|\sqrt{\mu\omega_0^2}}{4\sqrt{2}}\right)^{2/3}$$

Semiclassical approximation in the tails



$$\Psi_{SC}(x) = \begin{cases} C_{\text{left}} \left(\frac{8\pi S_{\text{left}}(x)}{3U_0} \right)^{1/2} \cos\left(\frac{\pi}{3}\right) \left[J_{1/3}(S_{\text{left}}(x)) + J_{1/3}(S_{\text{left}}(x)) \right] \\ 2C_{\text{right}} \left(\frac{8 S_{\text{right}}(x)}{\pi \left| U_0 \right|} \right)^{1/2} \cos\left(\frac{\pi}{3}\right) K_{1/3}(S_{\text{right}}(x)) \end{cases}$$

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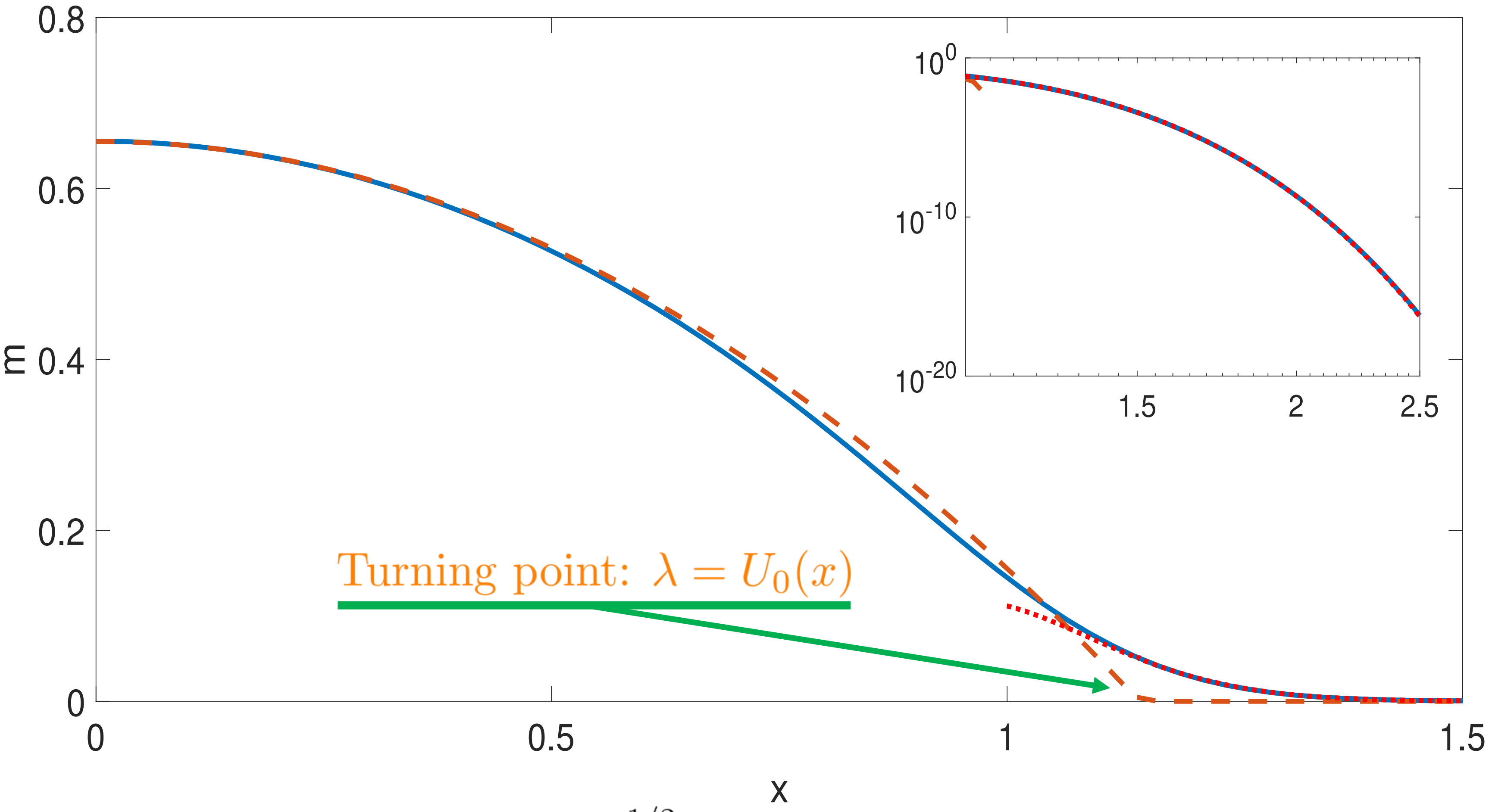

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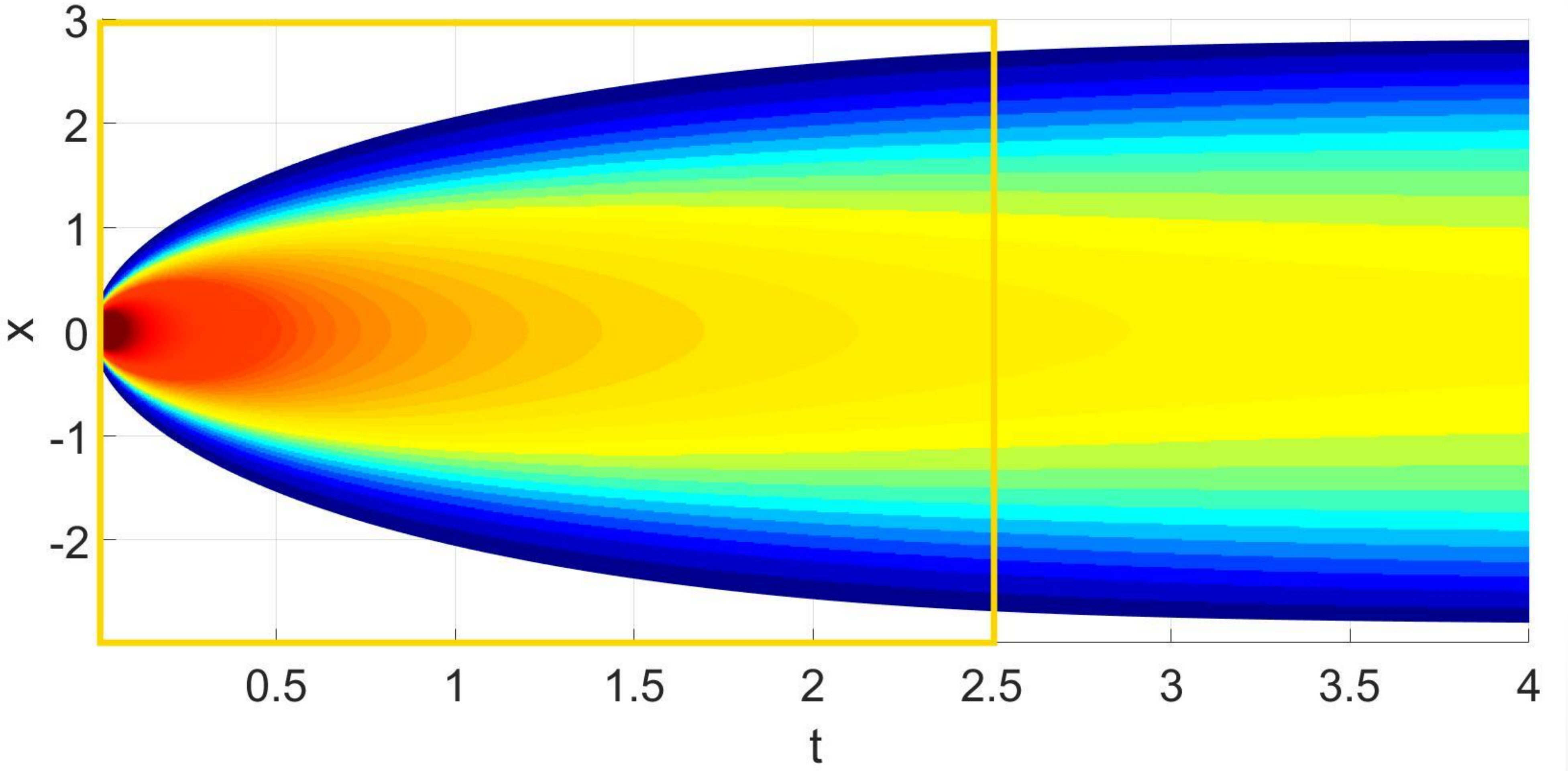
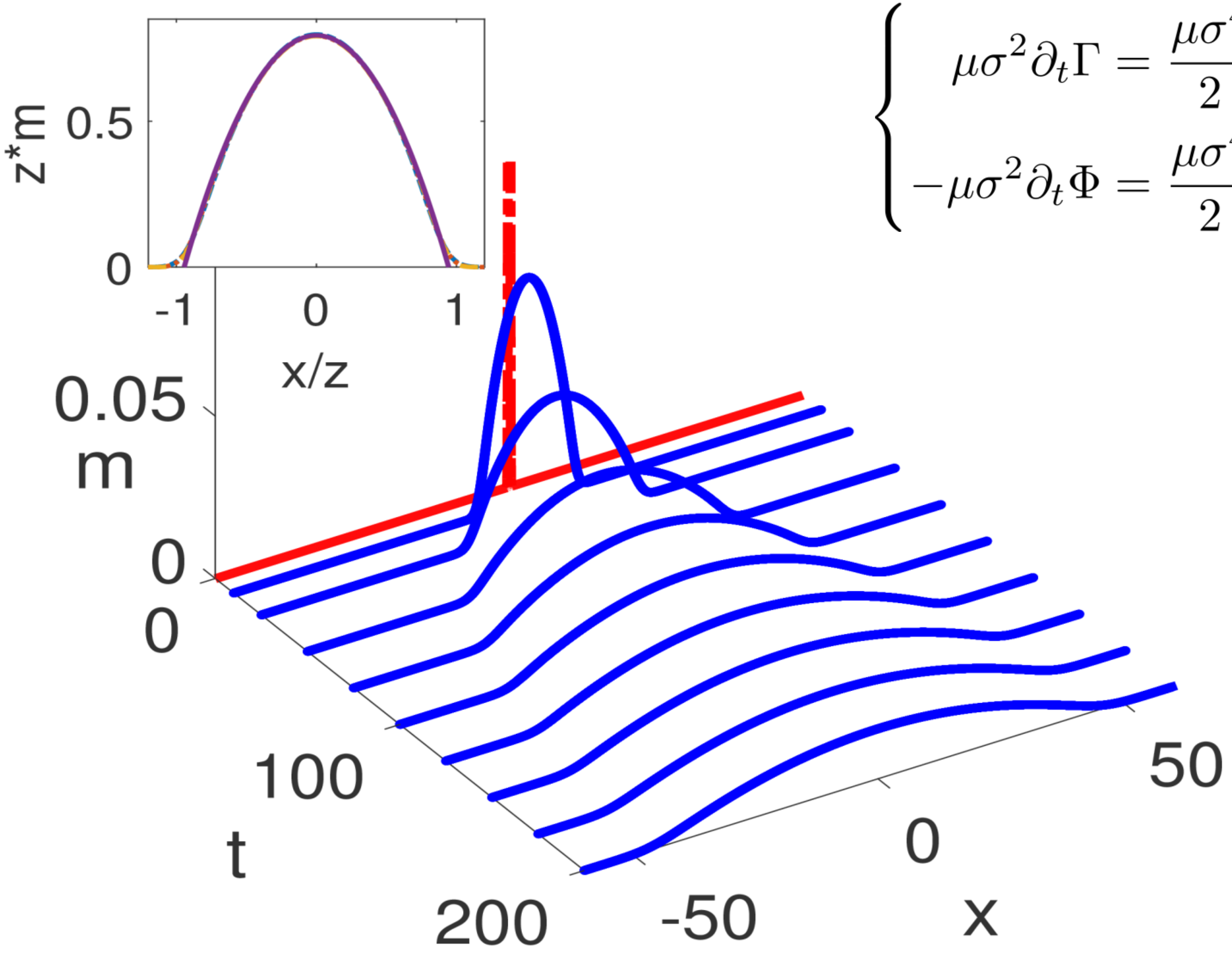
Beginning of the game



Long optimization time: numerical evidences



$$\begin{cases} \mu\sigma^2\partial_t\Gamma = \frac{\mu\sigma^4}{2}\Delta_{\mathbf{x}}\Gamma + g\,m\Gamma + U_0(\mathbf{x})\Gamma \\ -\mu\sigma^2\partial_t\Phi = \frac{\mu\sigma^4}{2}\Delta_{\mathbf{x}}\Phi + g\,m\Phi + U_0(\mathbf{x})\Phi \end{cases}$$

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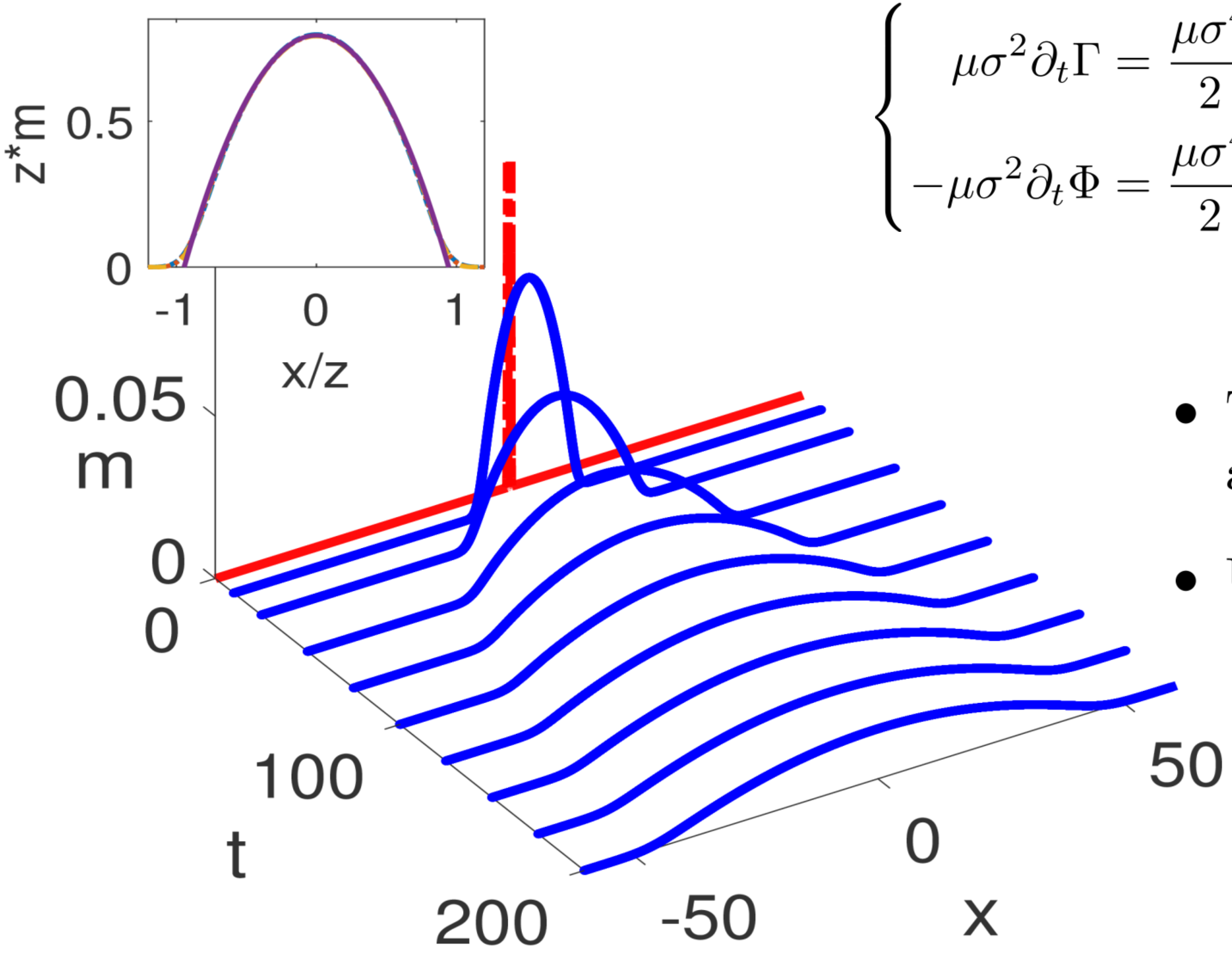

$$\begin{cases} \mu\sigma^2\partial_t\Gamma = \dfrac{\mu\sigma^4}{2}\Delta_{\mathbf{x}}\Gamma + g\,m\Gamma + \cancel{U_0(\mathbf{x})}\Gamma \\ -\mu\sigma^2\partial_t\Phi = \dfrac{\mu\sigma^4}{2}\Delta_{\mathbf{x}}\Phi + g\,m\Phi + \cancel{U_0(\mathbf{x})}\Phi \end{cases}$$

- The density spontaneously takes a parabolic shape
- Universal scaling behaviour as $t^{2/3}$

Hydrodynamic representation

- Madelung substitution $(\Phi(x,t), \Gamma(x,t)) \longrightarrow (m(x,t), v(x,t))$:

$$v(x,t) = \frac{\sigma^2}{2}\left(\frac{\nabla\Phi(x,t)}{\Phi(x,t)} - \frac{\nabla\Gamma(x,t)}{\Gamma(x,t)}\right)$$

$$\begin{cases} \partial_t m + \nabla(mv) = 0 \\ \partial_t v + \nabla\left[\frac{g}{\mu}m + \frac{v^2}{2} + \frac{\sigma^4}{2\sqrt{m}}\Delta\sqrt{m} + U_0(x)\right] = 0 \end{cases}$$

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- Energy

$$E = \int_{\mathbb{R}} \mathrm{d}x \left[\frac{\mu\sigma^2}{2}\left(m\left(\frac{v}{\sigma}\right)^2 - \sigma^2\frac{(\nabla m)^2}{4m}\right) + \frac{g}{2}m^2 + U_0 m\right]$$

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Hydrodynamic representation: parabolic ansatz

- Exact solution of hydrodynamic equations

$$\begin{cases} m = \dfrac{3\left[z(t)^2 - x^2\right]}{4z(t)^3} \\ v = -\dfrac{z'(t)}{z(t)}x \end{cases} \qquad z'(t)^2 = \frac{3|g|}{\mu}\left[\frac{1}{z(t)} + \frac{\epsilon}{\tilde{z}}\right]$$

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- For $\epsilon = -1$: negative energy

$$E = E^{\text{int}}(t = \tilde{T}) = \frac{3g}{10\tilde{z}} \quad \text{and} \quad \tilde{T} = \frac{\pi \tilde{z}^{3/2}}{2\sqrt{3|g|/\mu}}$$

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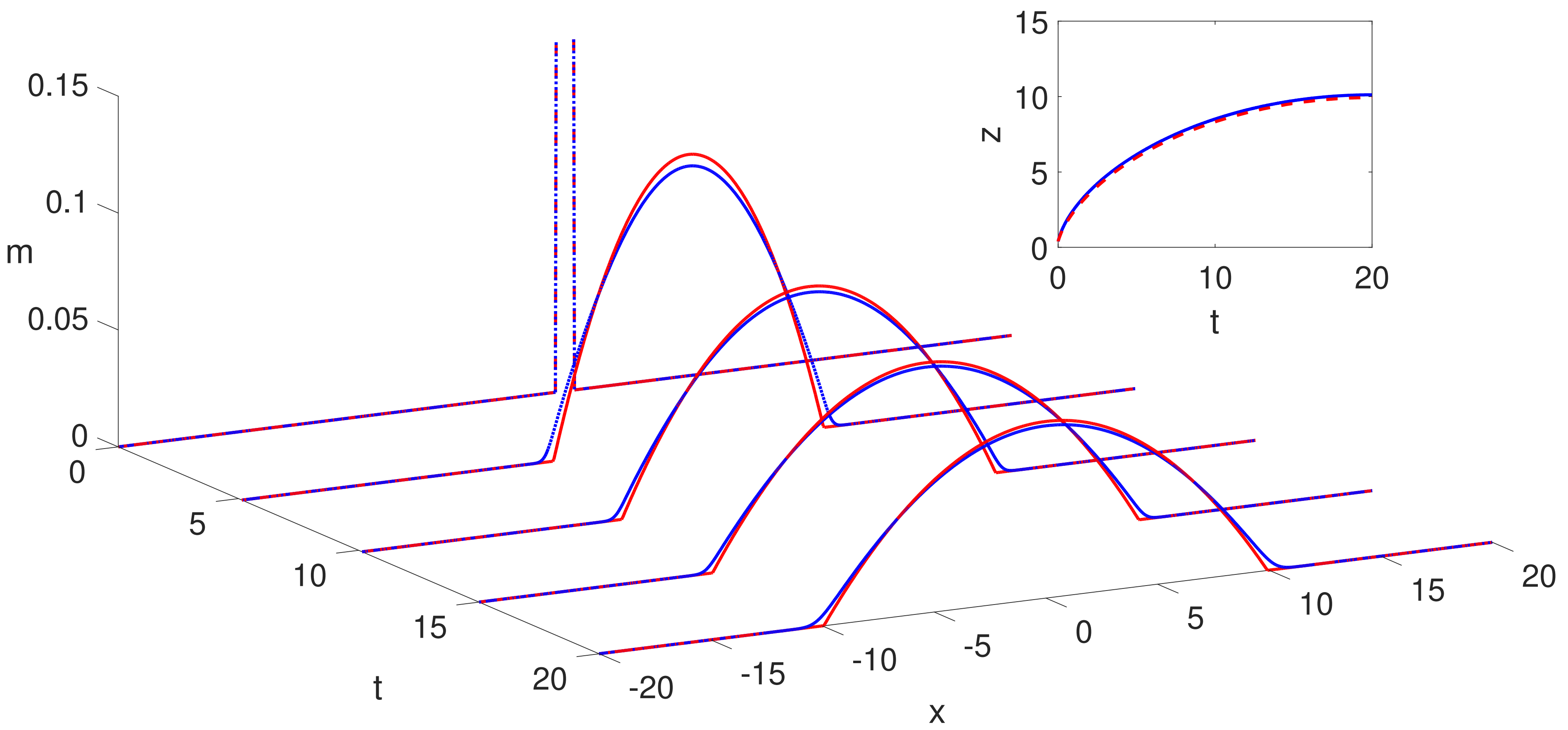

$$\frac{\sqrt{3|g|/\mu}}{\tilde{z}^{3/2}}(t-t_0) = \arcsin\sqrt{\frac{z(t)}{\tilde{z}}} - \sqrt{\frac{z(t)}{\tilde{z}}\left(1-\frac{z(t)}{\tilde{z}}\right)} \quad \Longrightarrow \quad z \sim t^{2/3}, \quad \text{if} \quad \frac{z}{\tilde{z}} \ll 1$$

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- Riemann invariants nomenclature

$$\lambda_\pm(x,t) = v(x,t) \pm 2i\sqrt{\frac{|g|m(x,t)}{\mu}}$$

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- Riemann invariants nomenclature

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Potential representation

- Hodograph transform: linearise the system

$$\Rightarrow \text{invert} \qquad (\eta(x,t), \xi(x,t)) \rightarrow (x(\eta,\xi), t(\eta,\xi))$$

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- Define a "field" $E(\eta,\xi)$ by analogy with electrostatics

$$\vec{E} = -\nabla V \; , \qquad \begin{cases} E_\eta = -\partial_\eta \chi = -\eta t \\ E_\xi = -\partial_\xi \chi = -2(x - \xi t) \end{cases}$$

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→ Linear equation we know how to solve!

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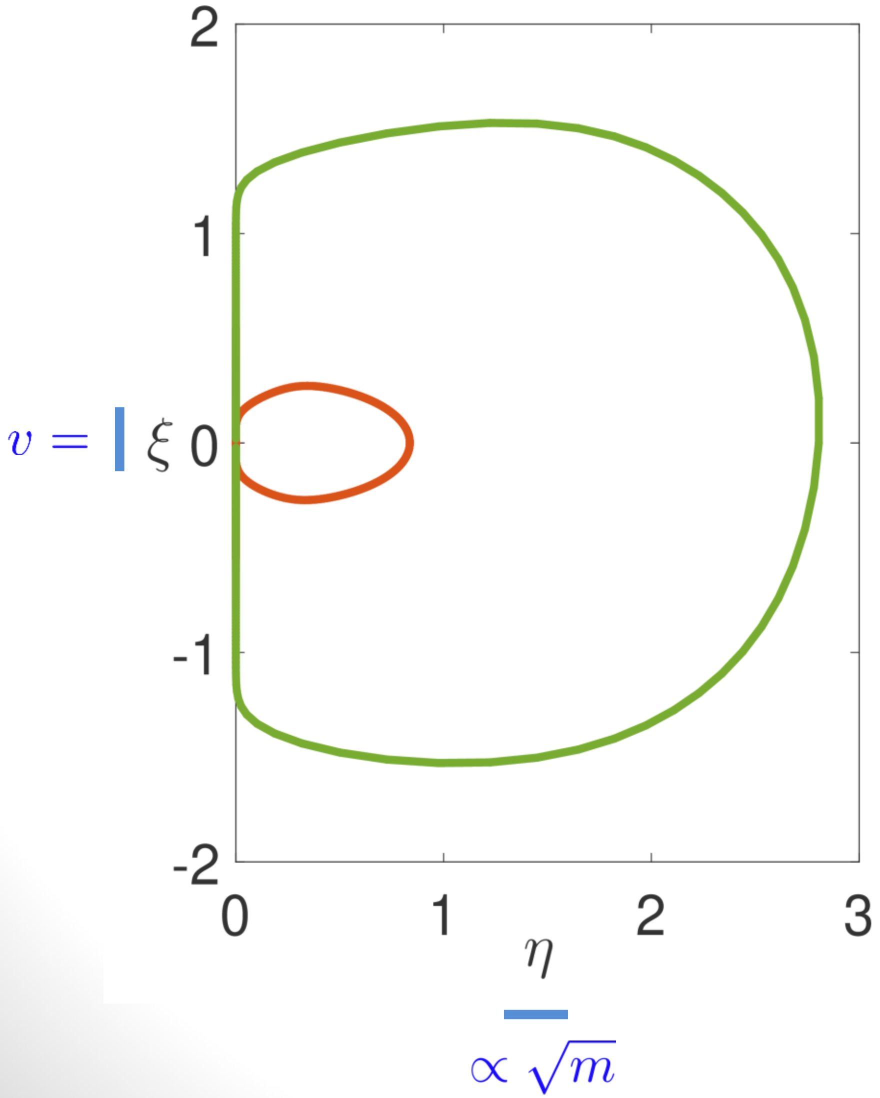
- Price to pay: implementing boundary conditions is not so straightforward

Multipole expansion



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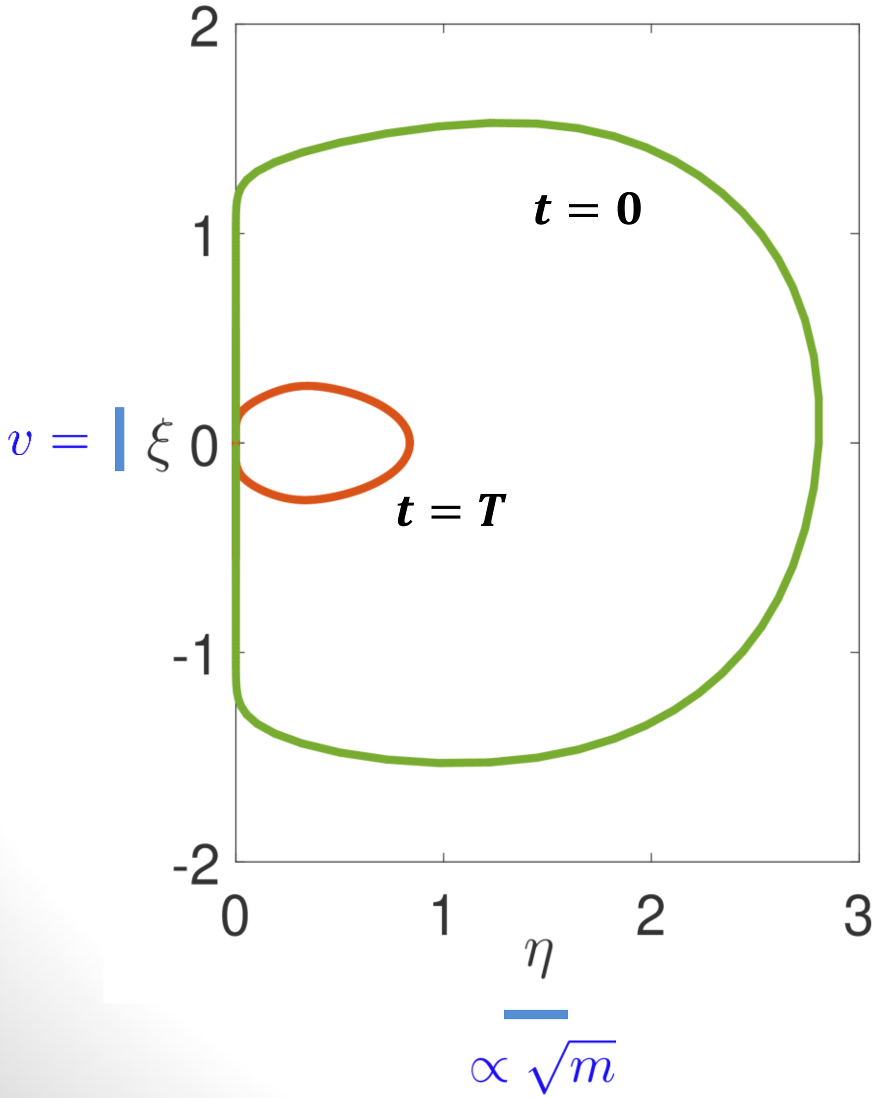

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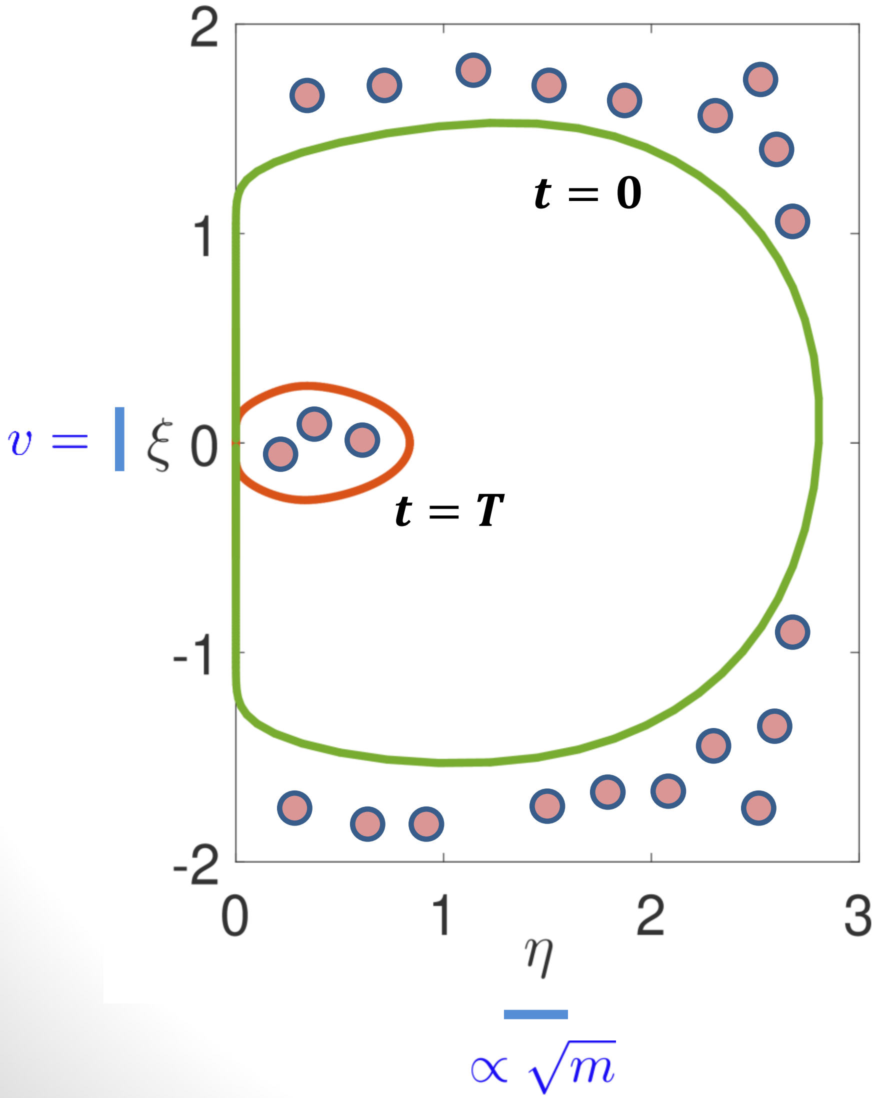

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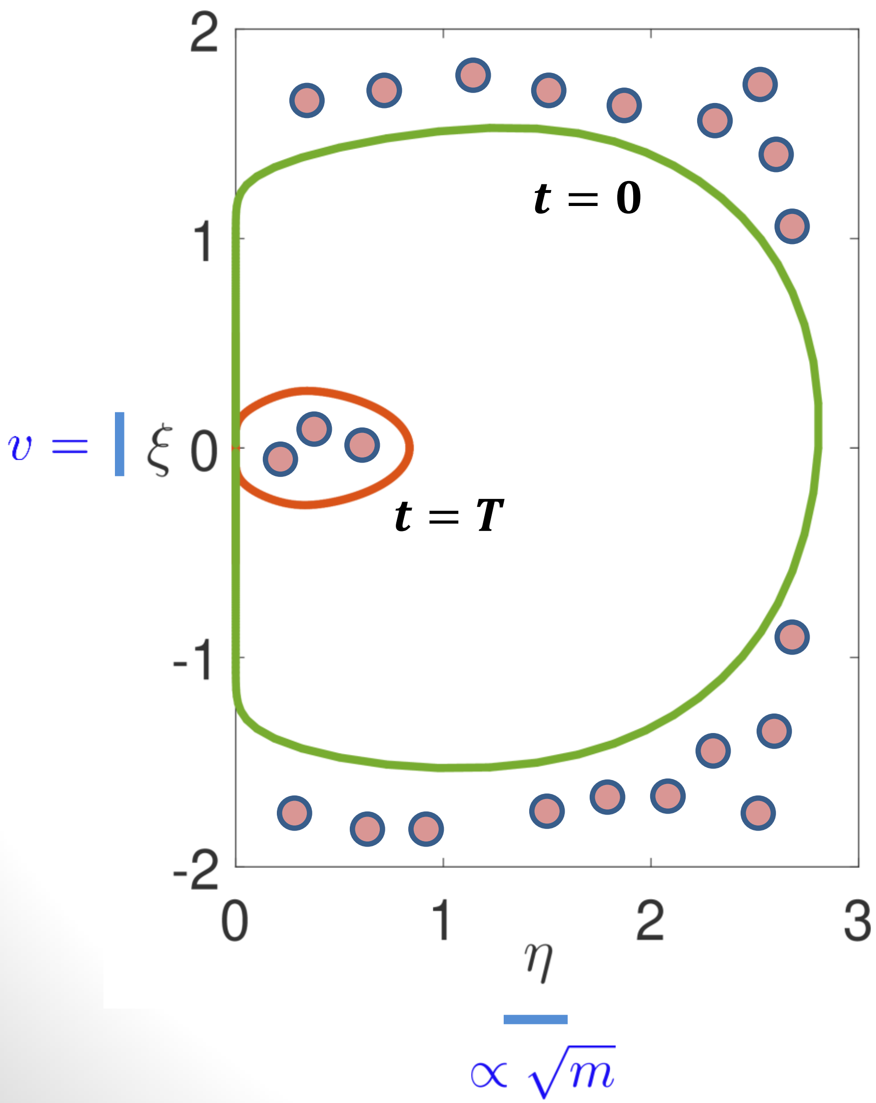

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- Multipole expansion ($r = \sqrt{\xi^2 + \eta^2}$)

$$\chi(\eta, \xi) = \sum_{l=0}^{\infty} \left(\frac{Q_l}{r^{l+1}} + I_l r^l \right) P_l \left(\frac{\xi}{r} \right)$$

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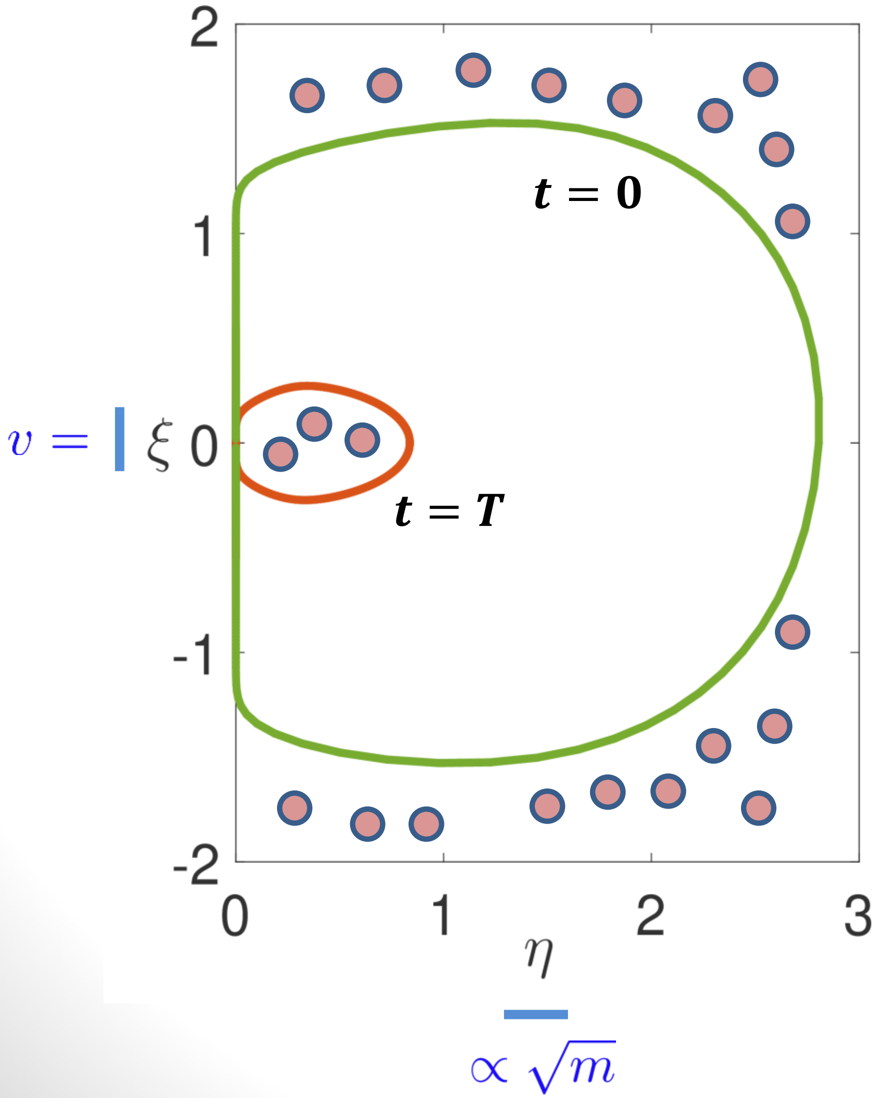

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- For $0 \ll t \ll T$: keep only the monopole

$$\chi(\eta, \xi) \approx \frac{Q_0}{r}$$

Long optimization time limit

- Monopole potential and field

$$\chi(\xi, \eta) = \frac{Q_0}{\sqrt{\eta^2 + \xi^2}}$$

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- Computing Q_0 using Gauss's law

$$Q_0 = \frac{1}{4\pi}\int_{S_{\tilde{t}}} (\vec{E}\cdot\vec{n})\mathrm{d}S = \frac{2g}{\mu}\overbrace{\int_{-\infty}^{+\infty} m(x)\mathrm{d}x}^{=1}$$

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χ is a generating function for the conservation laws.

Quadratic MFG and integrability

- Dimensionless form

$$\begin{cases} -\partial_{t'}\Phi = +\partial_{x'x'}\Phi + 2\,\epsilon\,\nu\,m\,\Phi \\ +\partial_{t'}\Gamma = +\partial_{x'x'}\Gamma + 2\,\epsilon\,\nu\,m\,\Gamma \end{cases} , \qquad x' = x/\nu \quad t' = t/\tau \qquad \nu = \frac{\mu\sigma^4}{|g|} \quad \tau = \frac{2\mu^2\sigma^6}{g^2}$$

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- Lax connection

$$\partial_t U - \partial_x V + [U, V] = 0$$

$$U = \kappa_\epsilon \begin{pmatrix} \frac{\lambda}{2} & \Phi \\ \Gamma & -\frac{\lambda}{2} \end{pmatrix} \quad , \qquad V = \kappa_\epsilon \begin{pmatrix} \kappa_\epsilon \Phi\Gamma & -\partial_x \Phi \\ \partial_x \Gamma & -\kappa_\epsilon \Phi\Gamma \end{pmatrix} - \lambda U$$

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- Poisson bracket and infinite hierearchy of conservation laws

$$\{F, G\} = \int_{\mathbb{R}} \left(\frac{\delta F}{\delta \Gamma}\frac{\delta G}{\delta \Phi} - \frac{\delta F}{\delta \Phi}\frac{\delta G}{\delta \Gamma} \right) \mathrm{d}x \quad , \qquad \{Q_n, Q_m\} = 0$$

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- Soliton solutions

$$\Psi(x,t) = 2b\mathrm{sech}\left[2b(x + 4at - x_0)\right] e^{\pm 2\left[ax + 2(a^2 - b^2)t + \phi_0\right]}$$

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Beginning of the game can be well approximated by an auxiliary game of flat terminal cost, zero external gain and energy E_e

Gaussian ansatz $\Rightarrow$ Parabolic ansatz $\Rightarrow$ Ergodic state

$\Sigma < \nu$ $\quad$ $z > \nu$ $\quad$ $z \approx z_e$

Relaxation towards the ergodic state

Evaluation of the duration τ of the transient state

$$\left.\begin{aligned} \tilde{T} &\sim \left[\tilde{E}^{\text{int}}\right]^{-3/2} \\ E_e &= \frac{3}{2} E_e^{\text{int}} \end{aligned}\right\} \quad \Rightarrow \quad \tau \approx \tilde{T}(3/2)^{3/2}$$