A Generalised Hydrodynamics approach to the Boussinesq equation: a prototypical example of 2D stationary soliton gas.

CIRM Workshop:
Emergent Hydrodynamics of Integrable
Systems and Soliton Gases

Thibault Bonnemain, 13th November 2023

[Based of joint work with G. Biondini, B. Doyon, G. El and G. Roberti]

The Boussinesq equation as a stationary reduction of KP

- KP equation: integrable nonlinear dispersive PDE in (2+1)D

$$\left[u_t + 6\left(u^2\right)_x + u_{xxx}\right]_x + \sigma u_{yy} = 0 \ ,$$

where case $\sigma = -1$ refered to as KPI and $\sigma = 1$ as KPII.

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- (boosted) KP equation: integrable nonlinear dispersive PDE in(2+1)D

$$\left[u_t - \sigma u_x + 6\left(u^2\right)_x + u_{xxx}\right]_x + \sigma u_{yy} = 0\ ,$$

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- Stationary solutions solve

$$\sigma\left(u_{yy} - u_{xx}\right) + 6\left(u^2\right)_x + u_{xxxx} = 0\ ,$$

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- Infinite set of conservation laws

$$\boxed{Q_n = \int \mathrm{d}x \ q_n(x,t) \ , \quad J_n = \int \mathrm{d}t \ j_n(x,t) \ , \quad \partial_t q_n + \partial_x j_n = 0 \ .}$$

N-soliton solution of KPII

[Freeman, Nimmo (1983)]

- $\tau-$function and Wronskian formulation: $u(x,y,t) = [\log \tau(x,y,t)]_{xx}$

$$\tau(x,y,t) = \mathrm{Wr}(f_1,\dots,f_N) = \det \begin{pmatrix} f_1 & f_2 & \cdots & f_N \\ f_1^{(1)} & f_2^{(1)} & \cdots & f_N^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(N-1)} & f_2^{(N-1)} & \cdots & f_N^{(N-1)} \end{pmatrix},$$

where the f_j's are linearly independent solutions of $f_y = \sqrt{3} f_{xx}$ and $f_t = f_x - 4 f_{xxx}$.

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- Soliton solutions

$$\begin{cases} f_i = \displaystyle\sum_{j=1}^{J} a_{i,j} e^{\theta_j} \\ \theta_j = k_j x + \sqrt{3} k_j^2 + (k_j - 4k_j^3)t + \theta_j^0 \end{cases}.$$

Properties of the solitons and their interactions

- Single soliton solution ($N = 1$ and $J = 2$)

$$u_1(x, y, t) = \frac{\eta^2}{2}\mathrm{sech}^2\left[\frac{\eta}{2}(x - cy - vt + x^0)\right] \ ,$$

with dispersion relation $v = \eta^2 + c^2 - 1$.

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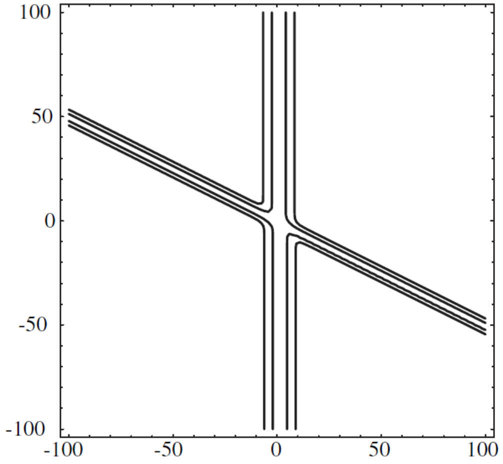
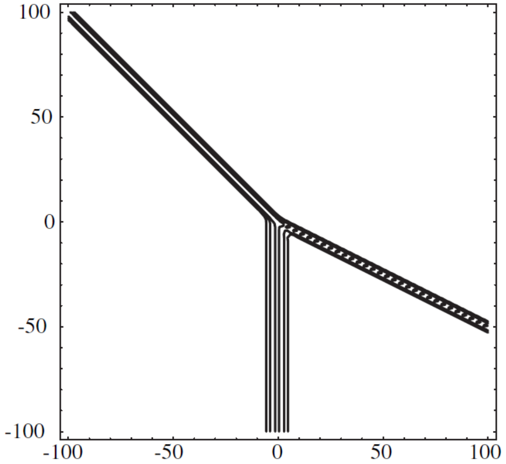
- Types of soliton interactions:

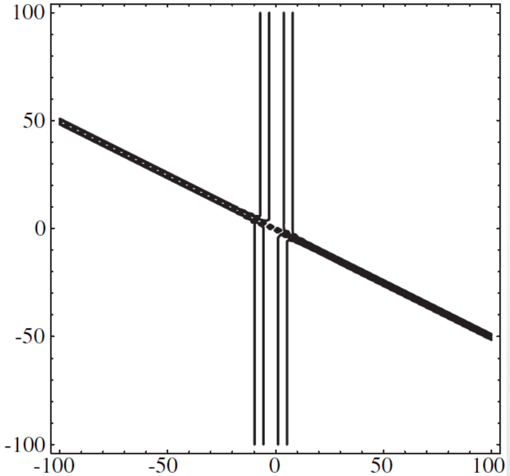
(i) Ordinary
$c_j - c_i < \sqrt{3}(\eta_i + \eta_j)$



(ii) Resonant
$\sqrt{3}(\eta_i + \eta_j) < c_j - c_i < \sqrt{3}(\eta_j - \eta_i)$



(iii) Anti-symmetric
$\sqrt{3}(\eta_j - \eta_i) < c_j - c_i$



[Biondini (2007)]

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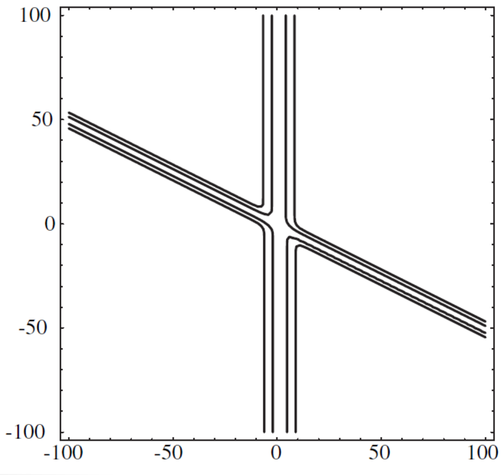

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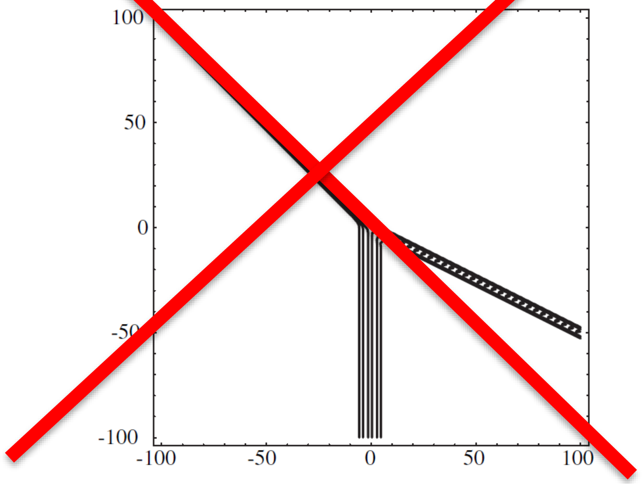

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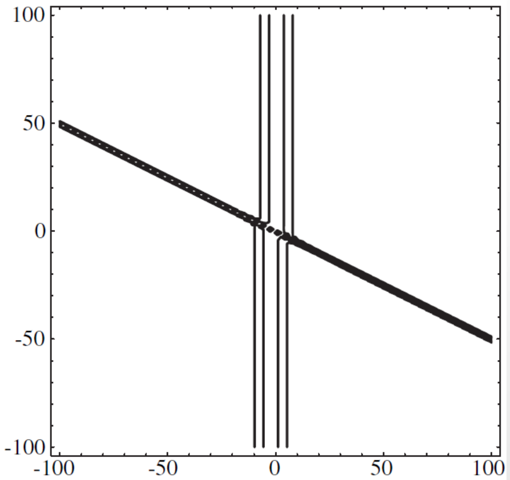
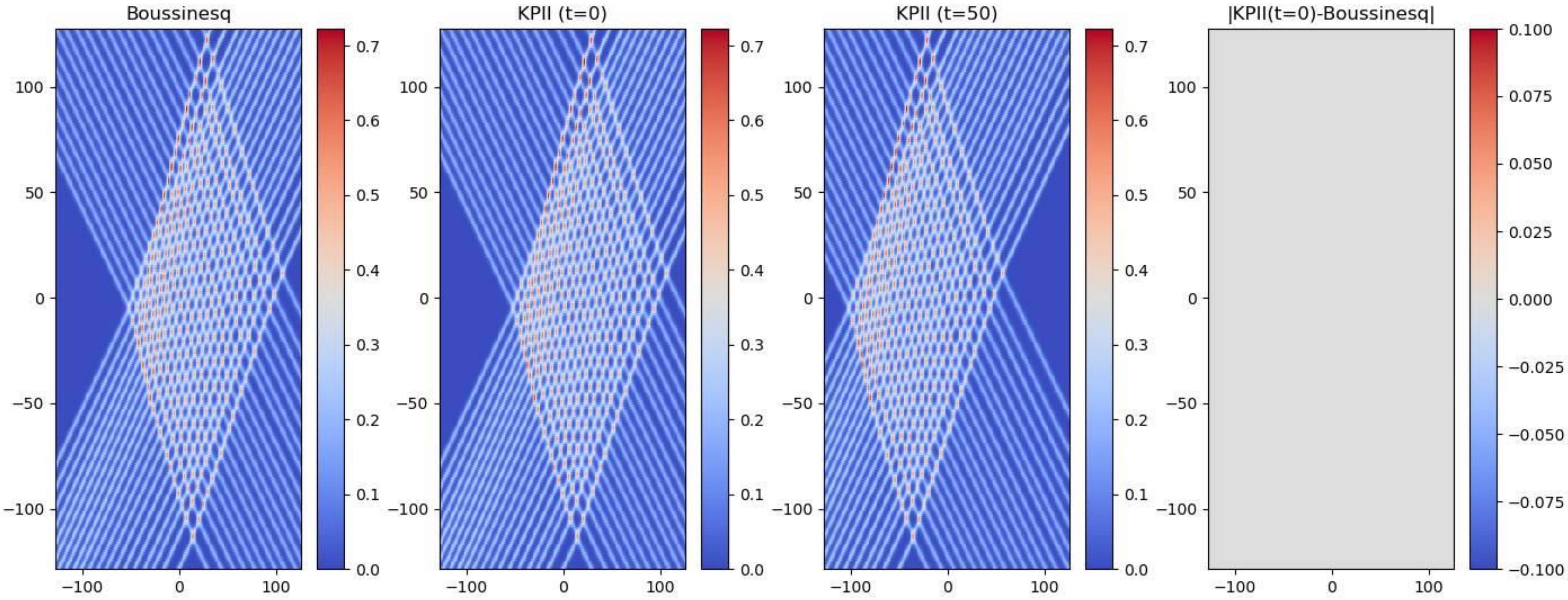


[Biondini (2007)]

Boussinesq vs (boosted) KP

[Courtesy of Giacomo Roberti]



N soliton solutions for the « good » Boussinesq equation

- $N-$soliton solution in terms of the $\tau-$function: $u_N(x,t) = [\log \tau(x,t)]_{xx}$

$$\tau(x,t) = 1 + \sum_{n=1}^{N} \sum_{{}_N C_n} a(i_1, i_2, \cdots, i_n) \exp\left[\theta_{i_1}(x,t) + \theta_{i_2}(x,t) + \cdots + \theta_{i_n}(x,t)\right] \ ,$$

with

$$\theta_j(x,t) = \eta_i \left(x - \epsilon_i t \sqrt{1-\eta_i^2} - x_i^0 \right) \ ,$$

$$a(i_1, i_2, \cdots, i_n) = \prod_{k<l}^{n} \exp \varphi_{i_k i_l} \ ,$$

$$\varphi_{ij} = \log \frac{\left(\epsilon_i \sqrt{1-\eta_i^2} - \epsilon_j \sqrt{1-\eta_j^2}\right)^2 - 3(\eta_i - \eta_j)^2}{\left(\epsilon_i \sqrt{1-\eta_i^2} - \epsilon_j \sqrt{1-\eta_j^2}\right)^2 - 3(\eta_i + \eta_j)^2} \ .$$

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φ_{ij}^{+} if $\epsilon_i \epsilon_j = 1$

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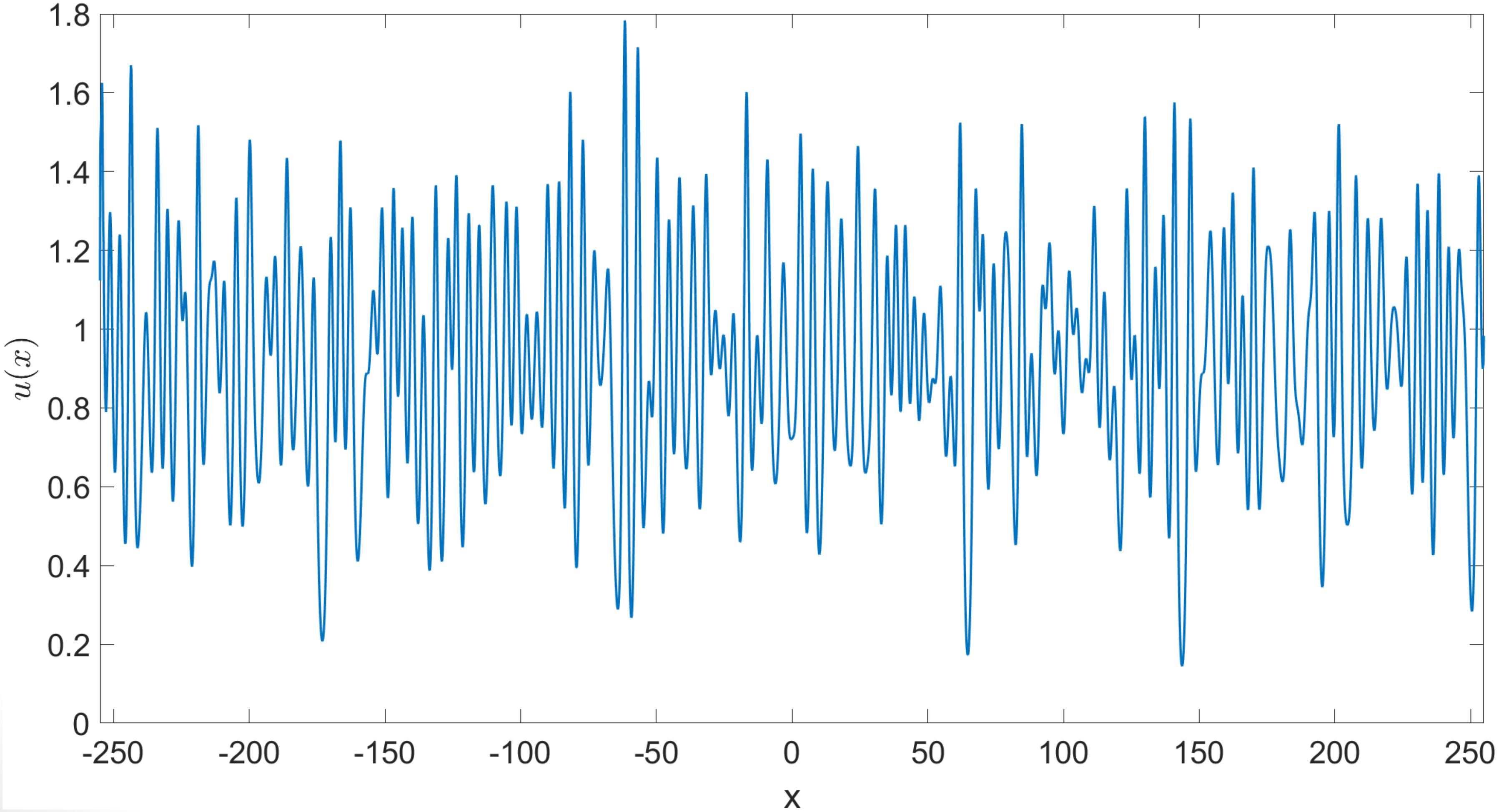
GHD from scattering theory

- Soliton gas: random solution to Boussinesq that can be well-described on a large interval by some $N-$soliton ensemble.



Single realisation of a soliton gas

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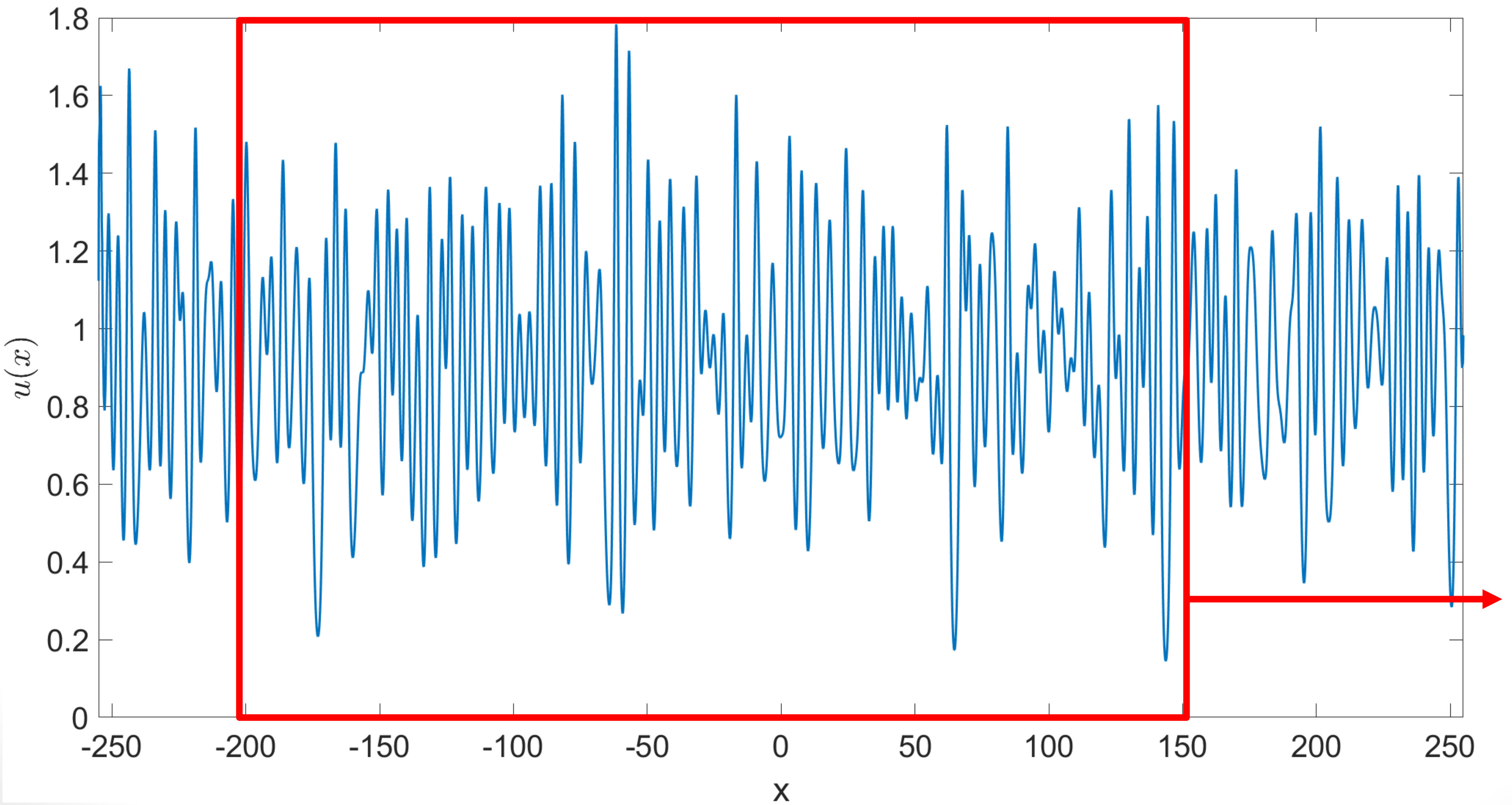

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Fluid cell of size L characterised by local GGE

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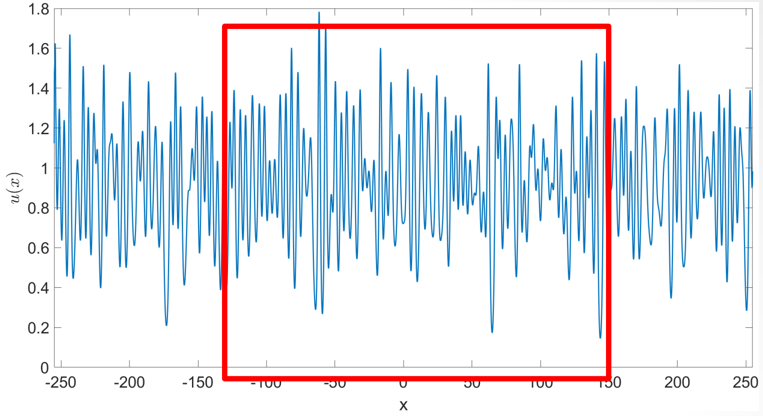
- Multi-soliton solution

$$u_N(x,t) \approx \sum_{i=1}^{N} \left(\frac{\eta_i}{2}\right)^2 \operatorname{sech}^2\left[\frac{\eta_i}{2}\left(x - \epsilon_i t\sqrt{1-\eta_i^2} - x_i^{\pm}\right)\right] \;, \quad \text{as } t \to \pm\infty$$



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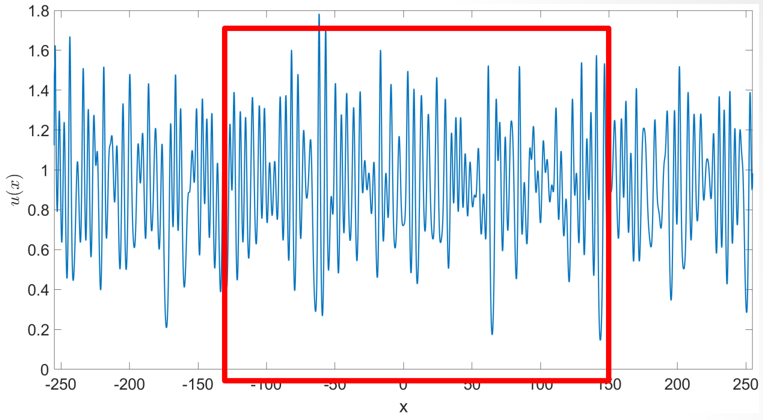

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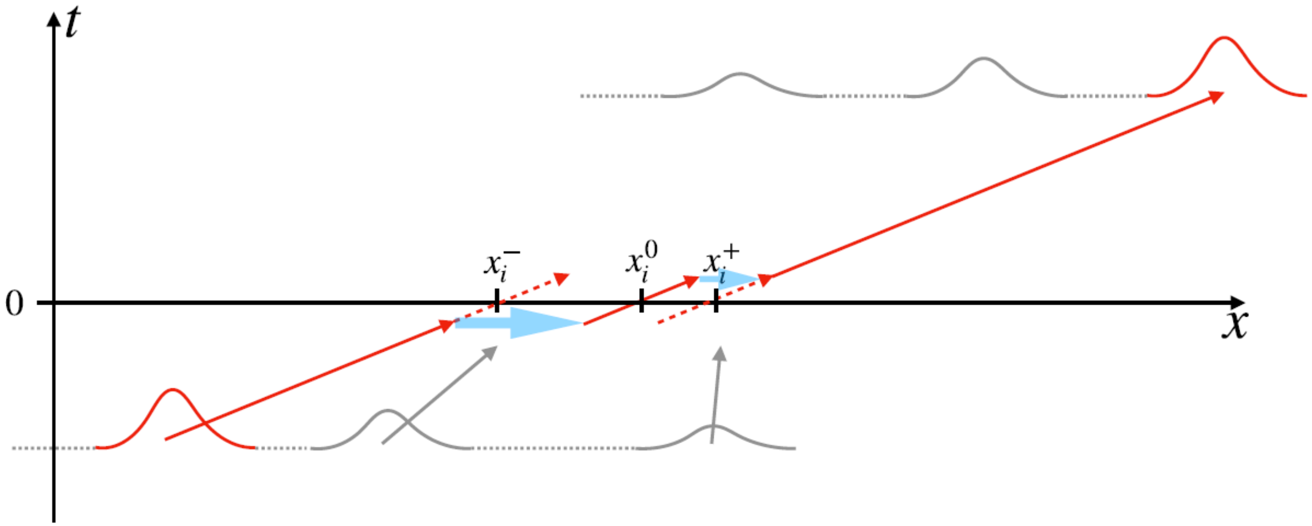
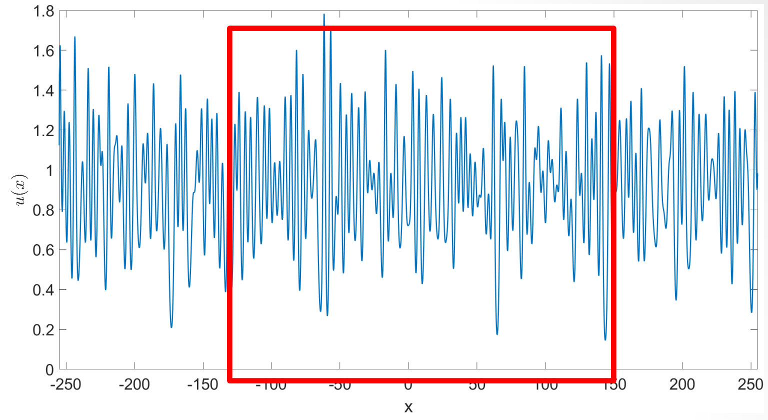


Scattering is elastic and
2-body factorisable

GHD from scattering theory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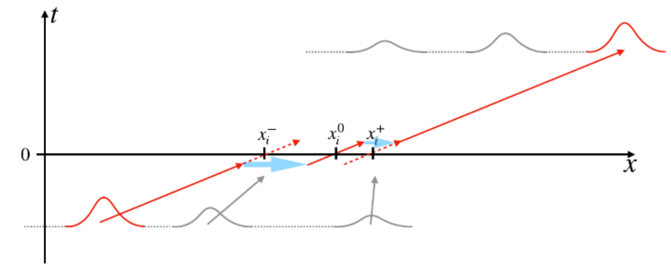
- Relation between asymptotic states given by scattering shift

$$x_i^+ - x_i^- = \sum_{j\neq i} \Delta_{ij} \ , \quad \Delta_{ij} = \begin{cases} \operatorname{sgn}(\eta_i - \eta_j)\dfrac{\varphi_{ij}^+}{\eta_i} & \text{if } \epsilon_i\epsilon_j = 1 \\ -\dfrac{\varphi_{ij}^-}{\epsilon_i\eta_i} & \text{if } \epsilon_i\epsilon_h = -1 \end{cases}$$



Thermodynamics

- $N-$soliton partition function can be formally written as

$$\mathcal{Z}_N = \int \mathcal{D}[u_N] \exp \left(\underbrace{S[u_N]}_{\text{Entropy}} - \underbrace{W[u_N]}_{\text{Generalised Gibbs weight}} \right) .$$

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Constraint / Entropy

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- Assume solitons are point particles of velocity v_i and position $X_i^t = x_i^t + v_i t$.

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- Let i be the leftmost soliton $(x_i^0 = 0)$

$$0 \quad = \quad x_i^{\text{left}} + \frac{1}{\eta_i} \sum_{j=i+1}^{N} \varphi_{ij}^{+} \ .$$

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$$\underbrace{0}_{\text{Position at t=0}} = \overbrace{x_i^{\text{left}}}^{\text{Asymptotic position } x_i^-} + \underbrace{\frac{1}{\eta_i}\sum_{j=i+1}^{N} \varphi_{ij}^{+}}_{\text{Shifts from faster solitons}} .$$

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- Let i be the rightmost soliton $(x_i^0 = L)$

$$L = x_i^{\text{right}} - \frac{1}{\eta_i} \left[\sum_{j=M+1}^{i-1} \varphi_{ij}^{+} + \sum_{j=1}^{M} \varphi_{ij}^{-} \right] \ .$$

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⇒

Asymptotic space in terms of real space

$$\begin{aligned} L_i^{\text{r}} &\equiv x_i^{\text{right}} - x_i^{\text{left}} \\ &= L + \frac{1}{\eta_i}\left[\sum_{j=1}^{M} \varphi_{ij}^{-} + \sum_{j=M+1,\ j\neq i}^{N} \varphi_{ij}^{+}\right] \end{aligned}$$

Asymptotic space density

- Let $L^{\mathrm{r}}_N(\eta)$ interpolate L^{r}_i

$$\mathcal{K}^{\mathrm{r}}_N(\eta) \equiv \frac{L^{\mathrm{r}}_N(\eta)}{L} = 1 + \frac{1}{L\eta}\left[\sum_{j=1}^{M} \varphi^{-}(\eta,\eta_j) + \sum_{j=M+1}^{N} \varphi^{+}(\eta,\eta_j)\right] .$$

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Aymptotic space density

Spectral density of states

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Asymptotic constraint

- $N-$soliton in asymptotic coordinates

$$\mathcal{Z}_N = \sum_{M=0}^{N-1} \frac{M!(N-M)!}{(N!)^2} \int_{\Gamma_{\mathrm{l}}^M \times \Gamma_{\mathrm{r}}^{N-M} \times \mathbb{R}^N} \prod_{i=1}^{N} \frac{\mathrm{d}v_i}{2\pi} \mathrm{d}x_i^- \\ \exp\left[-\sum_{i=1}^{N} w(\eta_i)\right] \chi\left(u_N(x, t=0) < \varepsilon_x,\ x \notin [0, L]\right) .$$

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- Asymptotic constraint

$$\int_{\mathbb{R}^N} \prod_{i=1}^{N} \mathrm{d}x_i^- \chi\left(u_N(x,t=0) < \varepsilon_x,\ x \notin [0,L]\right) \approx \prod_{i=1}^{N} \left(\int_{x_i^{\mathrm{left}}}^{x_i^{\mathrm{right}}} \mathrm{d}x^- \right)$$

$$= L^N \prod_{i=1}^{M} \mathcal{K}^{\mathrm{l}}(\eta_i) \prod_{i=M+1}^{N} \mathcal{K}^{\mathrm{r}}(\eta_i)\ .$$

Thermodynamic equilibrium

- Large deviations theory

[Varadhan (1966), Touchette (2009)]

$$\mathcal{Z}_N \asymp \exp\left(-L\mathcal{F}^{\mathrm{MF}}[\bar{\rho}^{\mathrm{l}}(\eta), \bar{\rho}^{\mathrm{r}}(\eta)]\right)$$

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$$\begin{aligned}\mathcal{F}^{\mathrm{MF}}[\rho_{\mathrm{l}}(\eta), \rho_{\mathrm{r}}(\eta)] &= \int_{\Gamma_{\mathrm{l}}} \mathrm{d}\eta \rho_{\mathrm{l}}(\eta)\left[w_{\mathrm{l}}(\eta) - 1 + \nu - \log\frac{\eta}{2\pi\sqrt{1-\eta^2}} - \log\mathcal{K}_{\mathrm{l}}(\eta) + \log\rho_{\mathrm{l}}(\eta)\right] \\ &+ \int_{\Gamma_{\mathrm{r}}} \mathrm{d}\eta \rho_{\mathrm{r}}(\eta)\left[w_{\mathrm{r}}(\eta) - 1 + \nu - \log\frac{\eta}{2\pi\sqrt{1-\eta^2}} - \log\mathcal{K}_{\mathrm{r}}(\eta) + \log\rho_{\mathrm{r}}(\eta)\right]\end{aligned}$$

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[Sanov (1961)]

$$\begin{aligned}\mathcal{F}^{\mathrm{MF}}[\rho_{\mathrm{l}}(\eta), \rho_{\mathrm{r}}(\eta)] &= \int_{\Gamma_{\mathrm{l}}} \mathrm{d}\eta\, \rho_{\mathrm{l}}(\eta) \left[\overbrace{w_{\mathrm{l}}(\eta)}^{\text{Gibbs weights}} - 1 + \nu - \overbrace{\log \frac{\eta}{2\pi\sqrt{1-\eta^2}}}^{\text{Jacobian}} - \log \mathcal{K}_{\mathrm{l}}(\eta) + \overbrace{\log \rho_{\mathrm{l}}(\eta)}^{\text{Configuration entropy}}\right] \\ &+ \int_{\Gamma_{\mathrm{r}}} \mathrm{d}\eta\, \rho_{\mathrm{r}}(\eta) \left[w_{\mathrm{r}}(\eta) \underbrace{- 1 + \nu}_{\text{Prefactor}} - \log \frac{\eta}{2\pi\sqrt{1-\eta^2}} \underbrace{- \log \mathcal{K}_{\mathrm{r}}(\eta)}_{\text{Constraint}} + \log \rho_{\mathrm{r}}(\eta)\right]\end{aligned}$$

$$\nu = \log\left[\gamma^{\gamma}(1-\gamma)^{1-\gamma}\right]$$

Yang-Yang type system

- Minimisation condition for the free energy functional

$$\begin{cases} \varepsilon_{\mathrm{l}}(\eta) = w_{\mathrm{l}}(\eta) + \nu + \log|v(\eta)| - \displaystyle\int_{\Gamma_{\mathrm{l}}} \frac{\mathrm{d}\mu}{2\pi}\, \varphi^{+}(\eta,\mu) e^{-\varepsilon_{\mathrm{l}}(\mu)} - \int_{\Gamma_{\mathrm{r}}} \frac{\mathrm{d}\mu}{2\pi} \varphi^{-}(\eta,\mu) e^{-\varepsilon_{\mathrm{r}}(\mu)} \\ \varepsilon_{\mathrm{r}}(\eta) = w_{\mathrm{r}}(\eta) + \nu + \log|v(\eta)| - \displaystyle\int_{\Gamma_{\mathrm{r}}} \frac{\mathrm{d}\mu}{2\pi}\, \varphi^{+}(\eta,\mu) e^{-\varepsilon_{\mathrm{r}}(\mu)} - \int_{\Gamma_{\mathrm{l}}} \frac{\mathrm{d}\mu}{2\pi} \varphi^{-}(\eta,\mu) e^{-\varepsilon_{\mathrm{l}}(\mu)} \end{cases} .$$

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- Integrability: infinite number of conservation laws

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$$\bar{j}_n(x,t) = \int_{\Gamma_\mathrm{l}} \mathrm{d}\eta\ \rho_\mathrm{l}(\eta;x,t) h^\mathrm{l}_n(\eta) v^\mathrm{eff}_\mathrm{l}(\eta;x,t) + \int_{\Gamma_\mathrm{r}} \mathrm{d}\eta\ \rho_\mathrm{r}(\eta;x,t) h^\mathrm{r}_n(\eta) v^\mathrm{eff}_\mathrm{r}(\eta;x,t)$$

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[Based on: Doyon, Spohn, Yoshimura (2017)]

- Asymptotic dynamics

$$x_i^-(t) = x_i^-(0) + v(\eta_i)t$$

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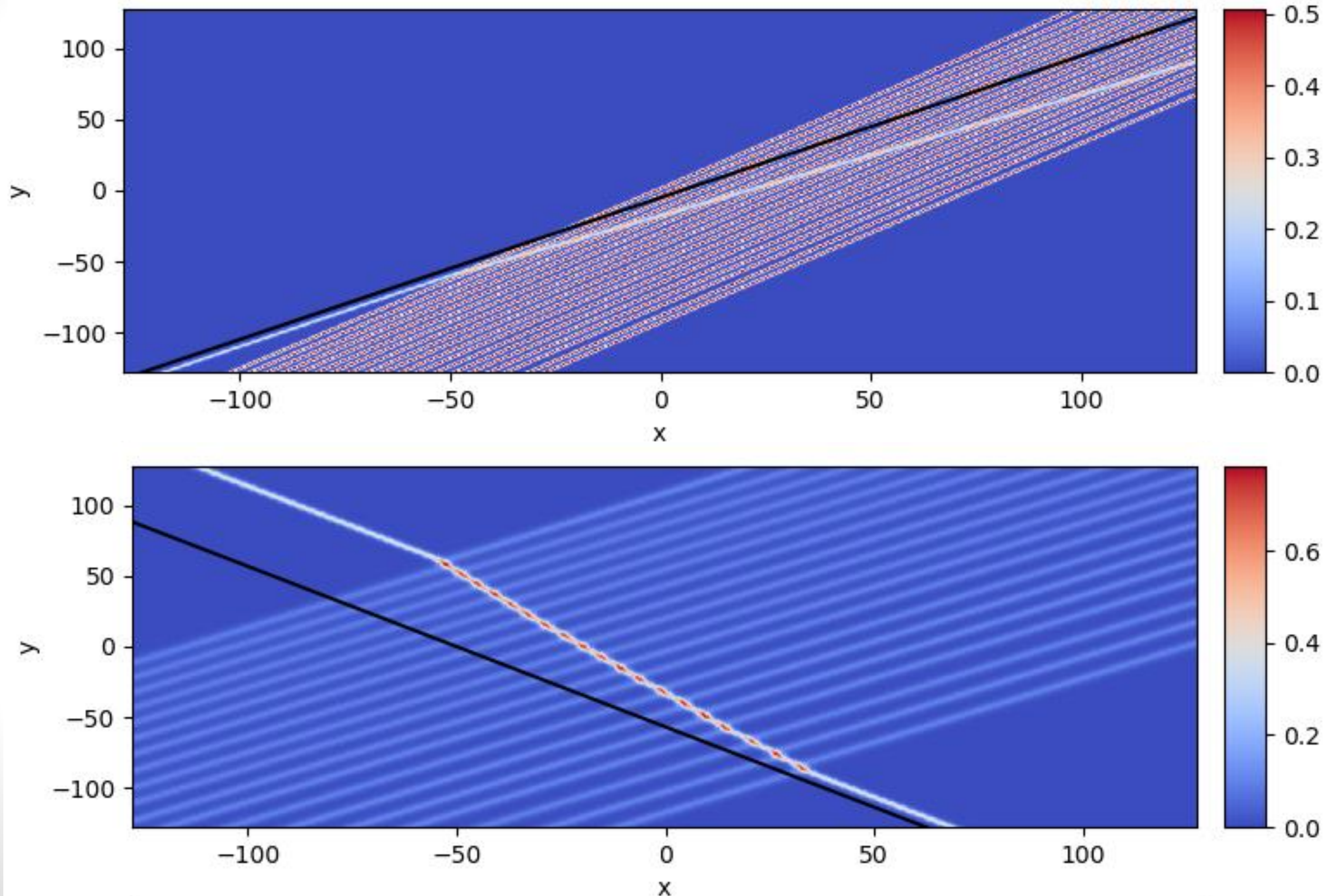
- Continuity equation for the DOS's

$$\boxed{\partial_t \rho_.(\eta; x, t) + \partial_x\left[\rho_.(\eta; x, t)v^{\mathrm{eff}}_.(\eta; x, t)\right] = 0}$$

Simulations

[Courtesy of Giacomo Roberti]



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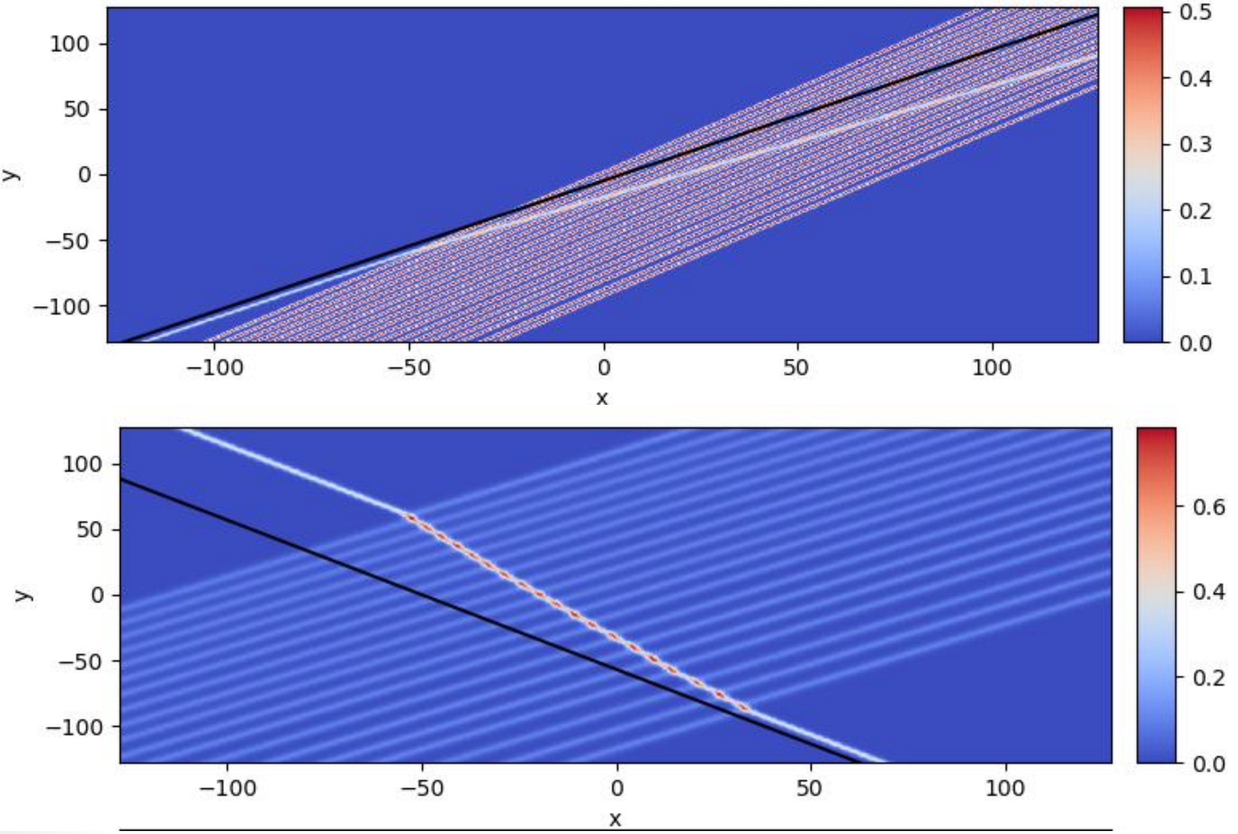

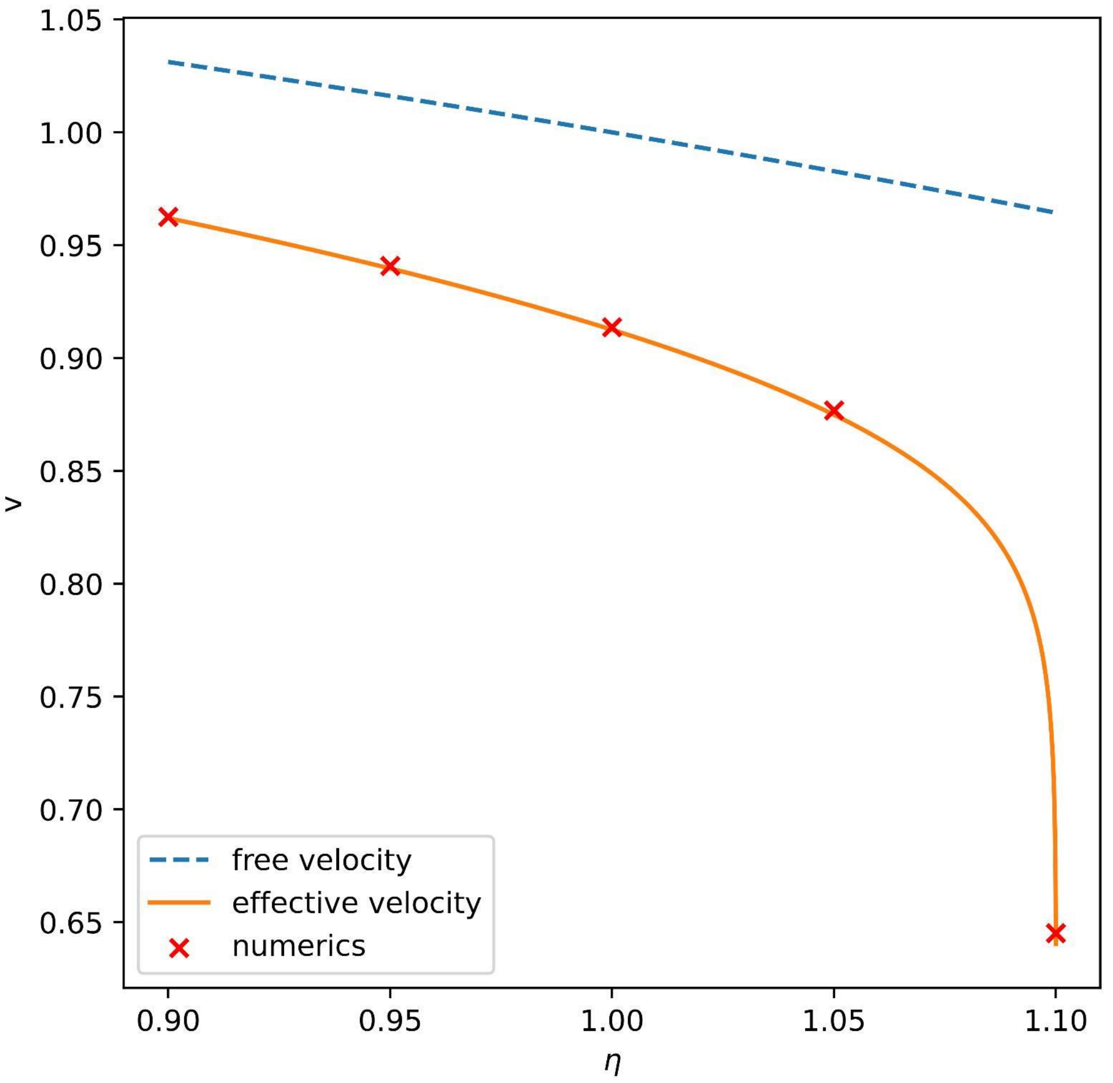

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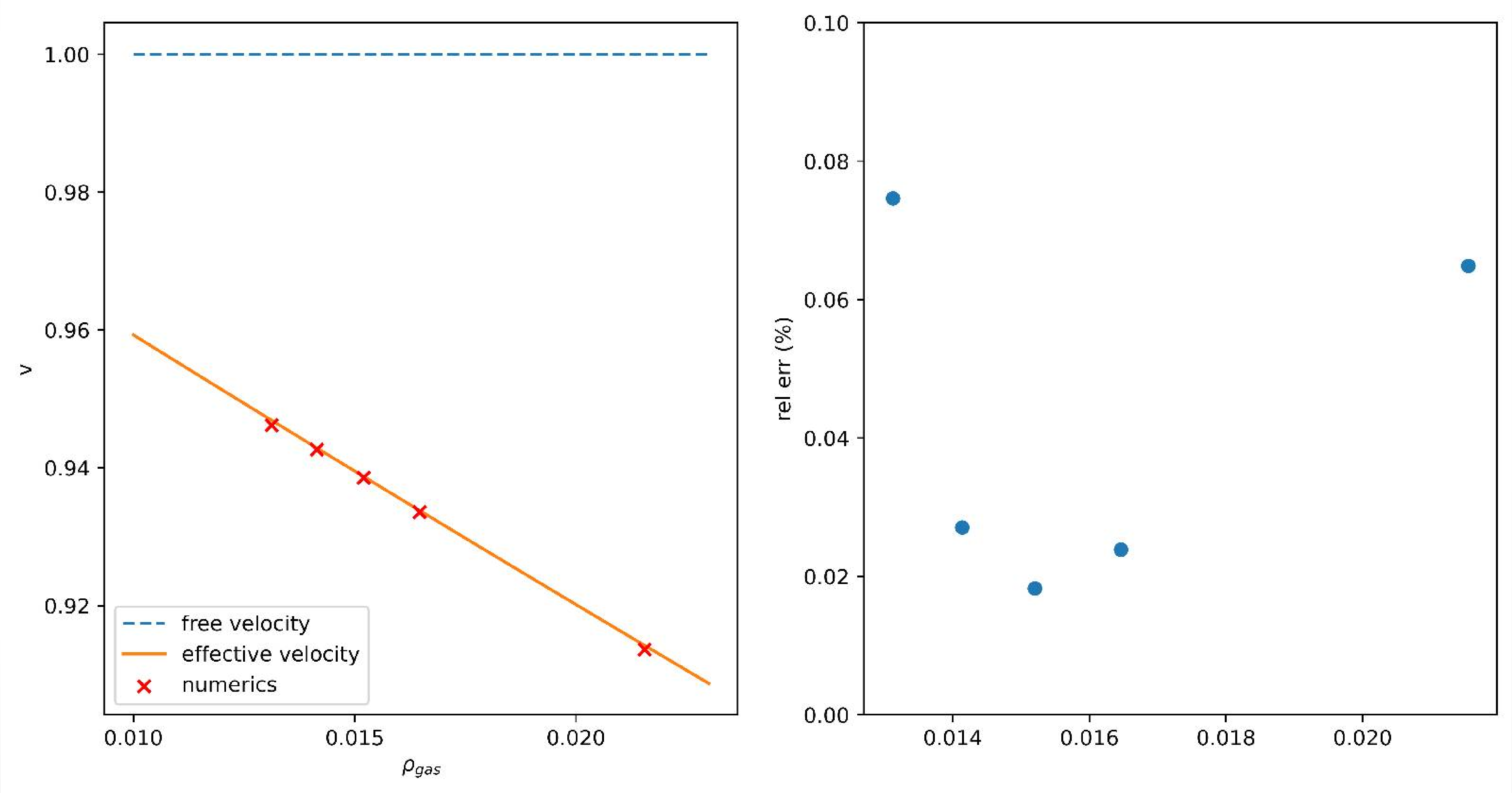

Concluding remarks and perspectives

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- A priori KP resonant interactions are unaccessible.
- Generic way to study integrable models in (d+1)D that involve "solitons" of co-dimension 1?