A Mean-Field Game theoretical approach to crowd dynamics

University of Bristol

Thibault Bonnemain, 18th October 2023

[Based on works w/ D. Ullmo, T. Gobron, M. Butano, C. Appert-Rolland, I. Echeverria-Huarte A. Nicolas, A. Seguin]

Pedestrian behavioral levels

[Hoogendoorn, Bovy (2004)]

Pedestrian behavioral levels

[Hoogendoorn, Bovy (2004)]

- Strategic level: Where (and when) should I go? What should I do?

Pedestrian behavioral levels

[Hoogendoorn, Bovy (2004)]

- Strategic level: Where (and when) should I go? What should I do?
- Tactical level: What route should I take? How should I schedule my activities?

Pedestrian behavioral levels

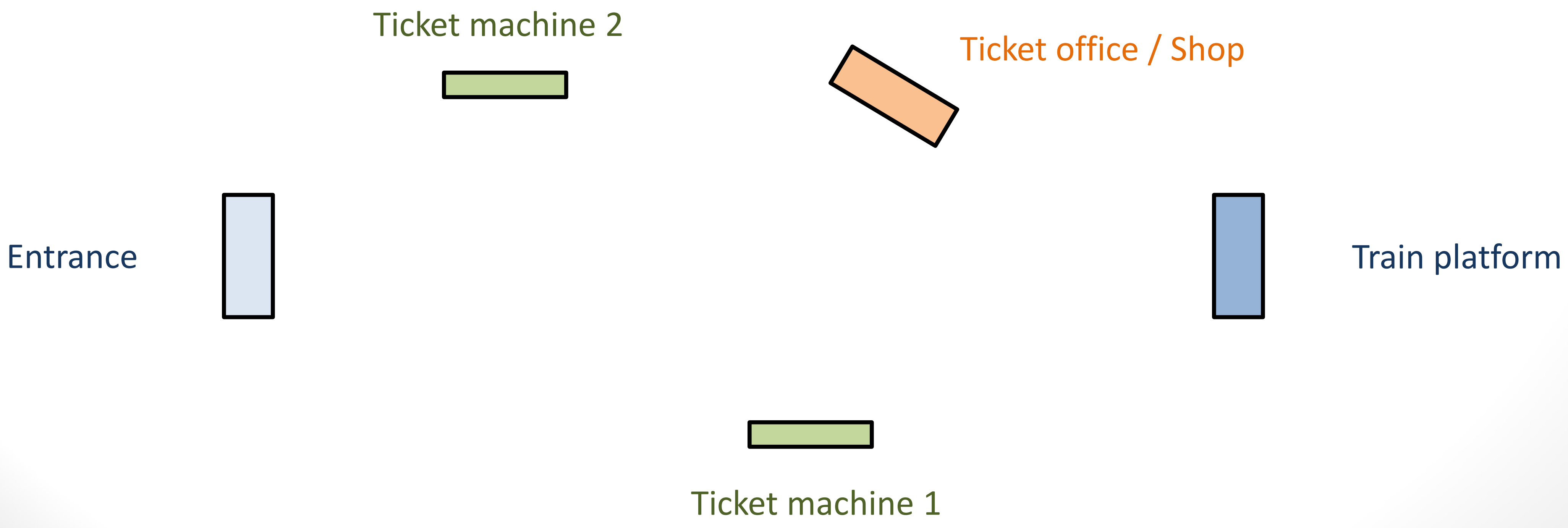
[Hoogendoorn, Bovy (2004)]

- Strategic level: Where (and when) should I go? What should I do?
- Tactical level: What route should I take? How should I schedule my activities?
- Operational level: How do I interact with the environment locally, en route?

Pedestrian behavioral levels

[Hoogendoorn, Bovy (2004)]

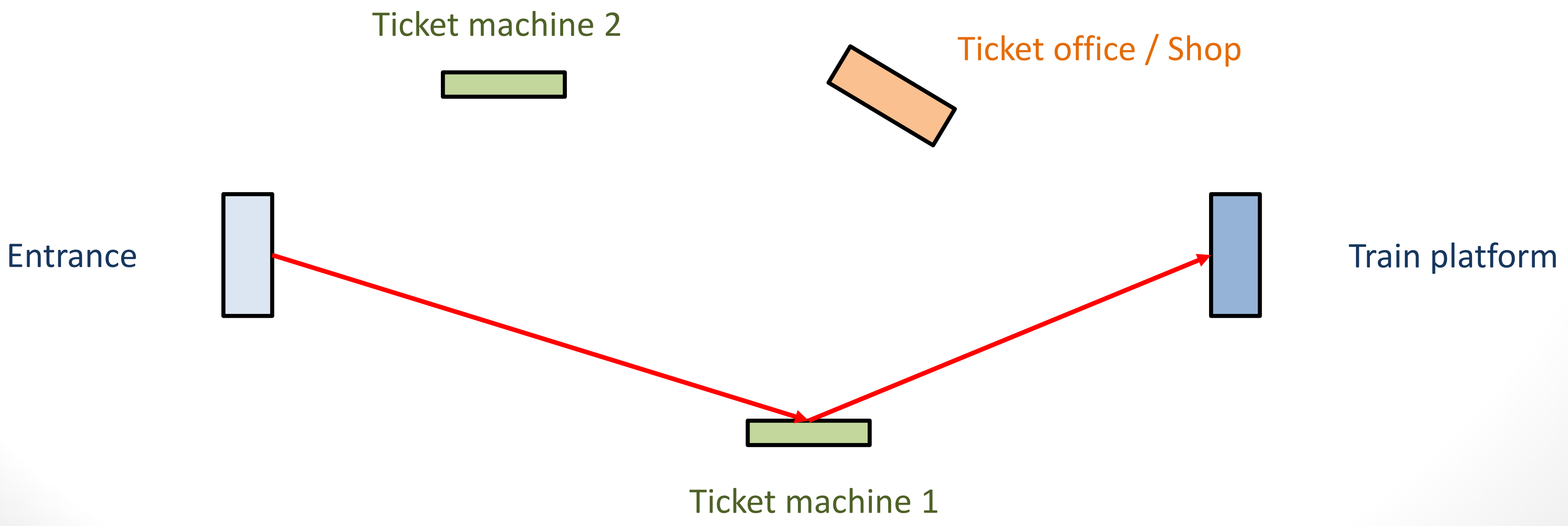
- Strategic level: Where (and when) should I go? What should I do?
- Tactical level: What route should I take? How should I schedule my activities?
- Operational level: How do I interact with the environment locally, en route?



Pedestrian behavioral levels

[Hoogendoorn, Bovy (2004)]

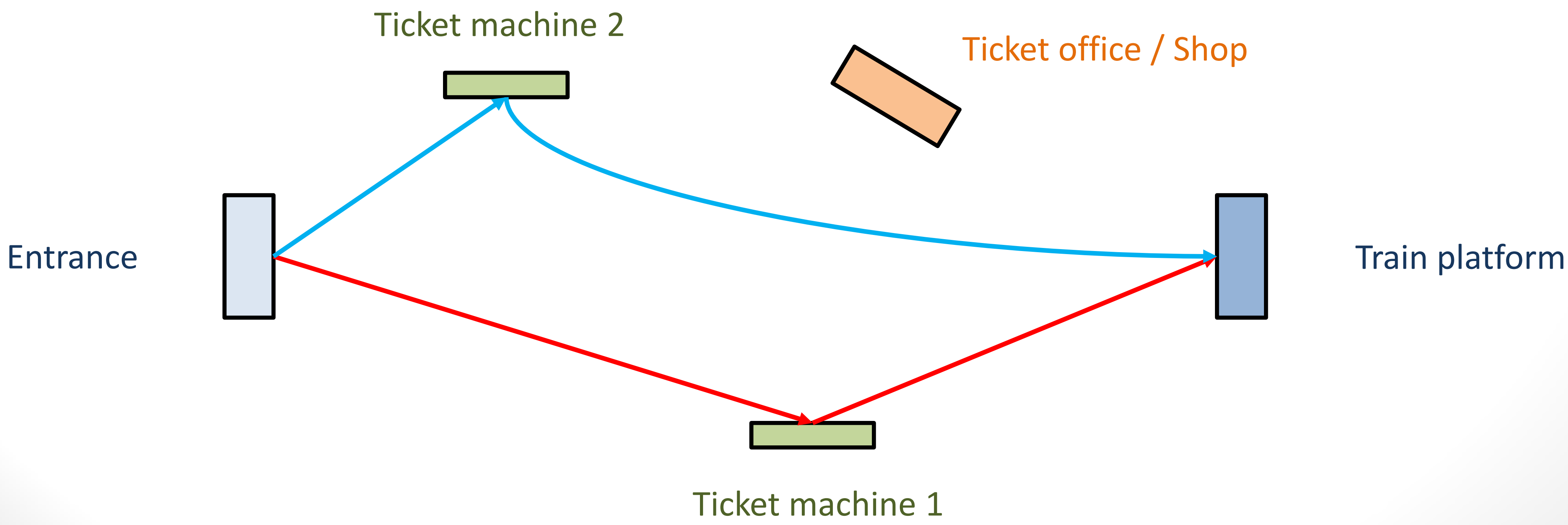
- Strategic level: Where (and when) should I go? What should I do?
- Tactical level: What route should I take? How should I schedule my activities?
- Operational level: How do I interact with the environment locally, en route?



Pedestrian behavioral levels

[Hoogendoorn, Bovy (2004)]

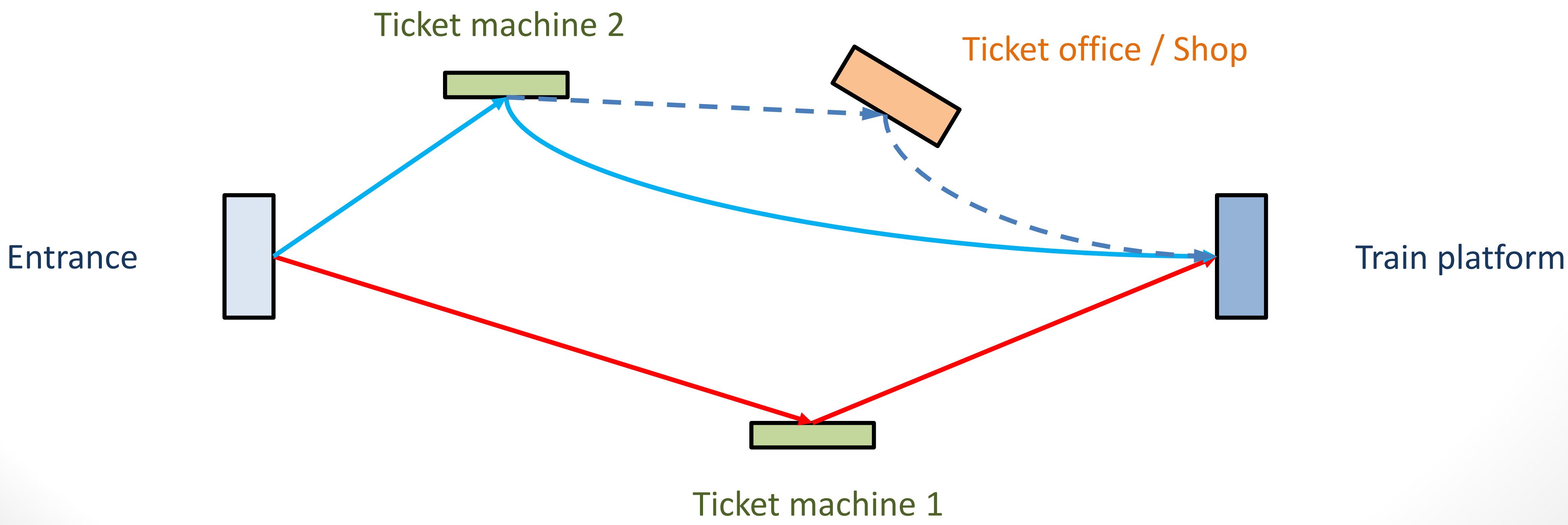
- Strategic level: Where (and when) should I go? What should I do?
- Tactical level: What route should I take? How should I schedule my activities?
- Operational level: How do I interact with the environment locally, en route?



Pedestrian behavioral levels

[Hoogendoorn, Bovy (2004)]

- Strategic level: Where (and when) should I go? What should I do?
- Tactical level: What route should I take? How should I schedule my activities?
- Operational level: How do I interact with the environment locally, en route?



Intruder crossing a static crowd

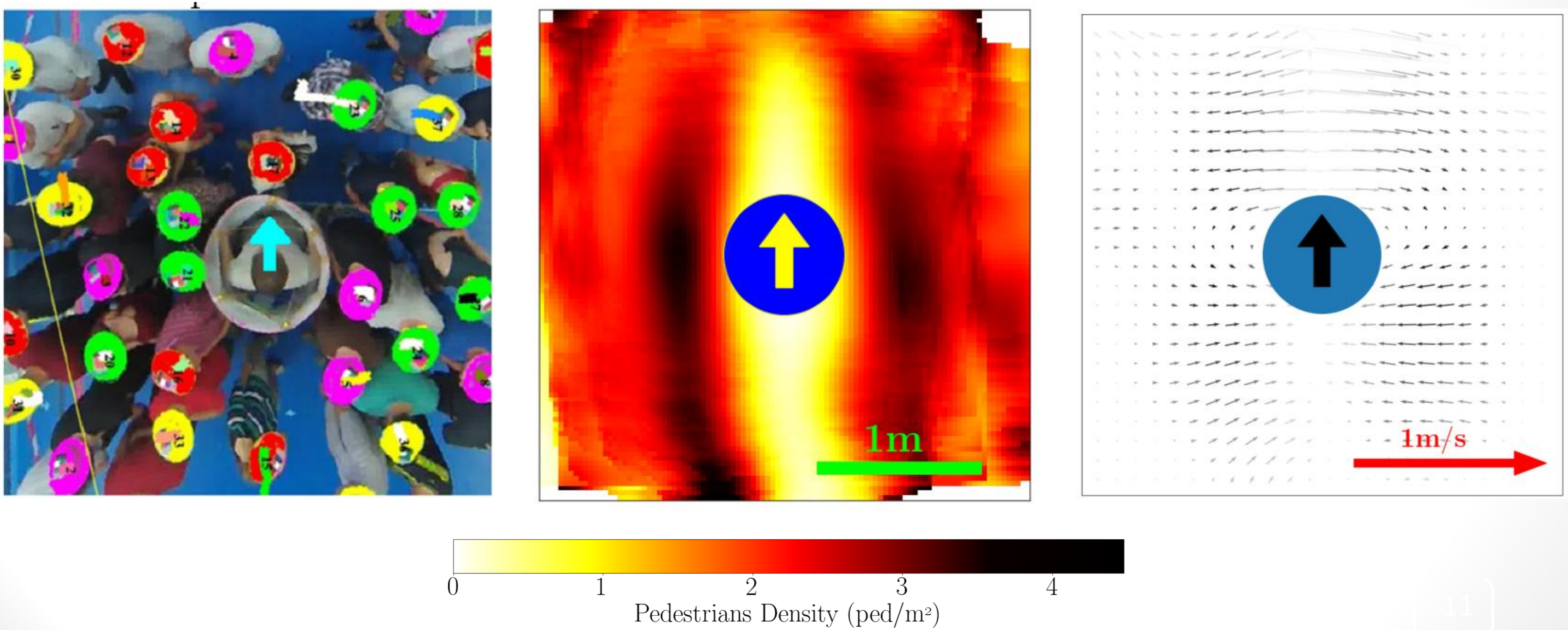
[Nicolas et al. (2019)]

Intruder crossing a static crowd

Experiment

(Averaged) pedestrian density

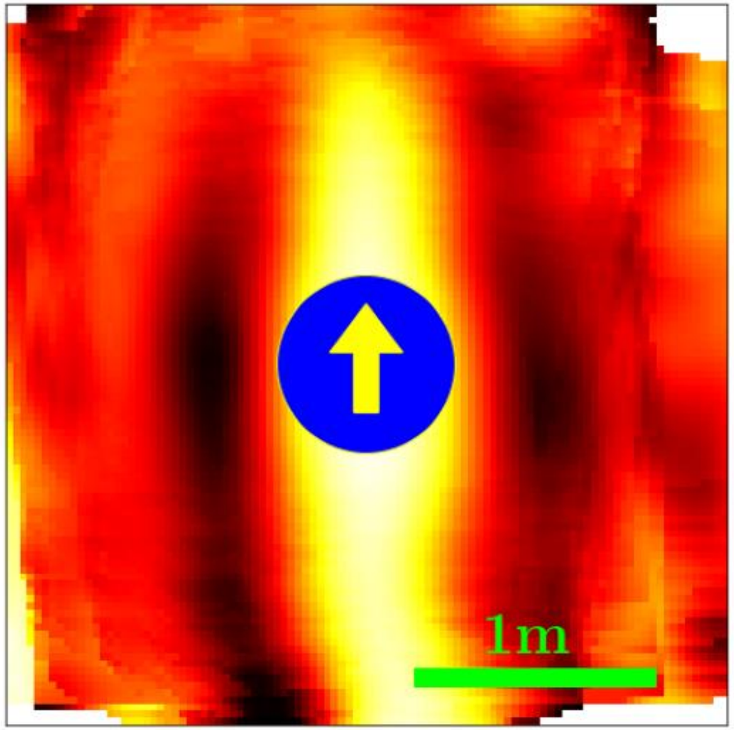
Velocity field



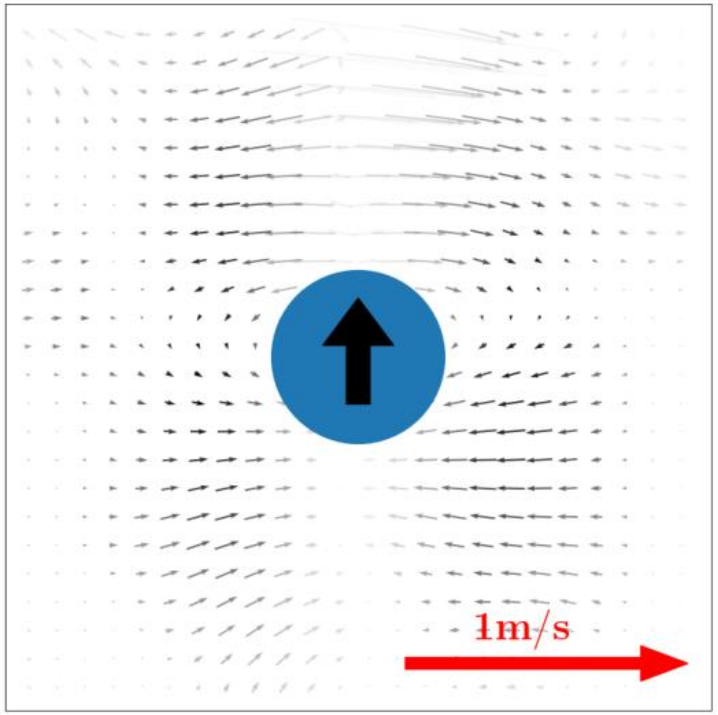
Intruder crossing a static crowd

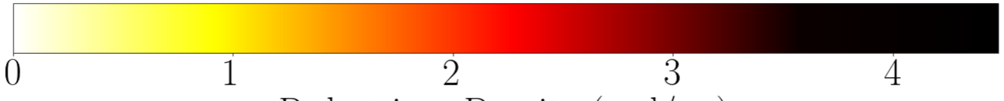
Experiment

(Averaged) pedestrian density



Velocity field



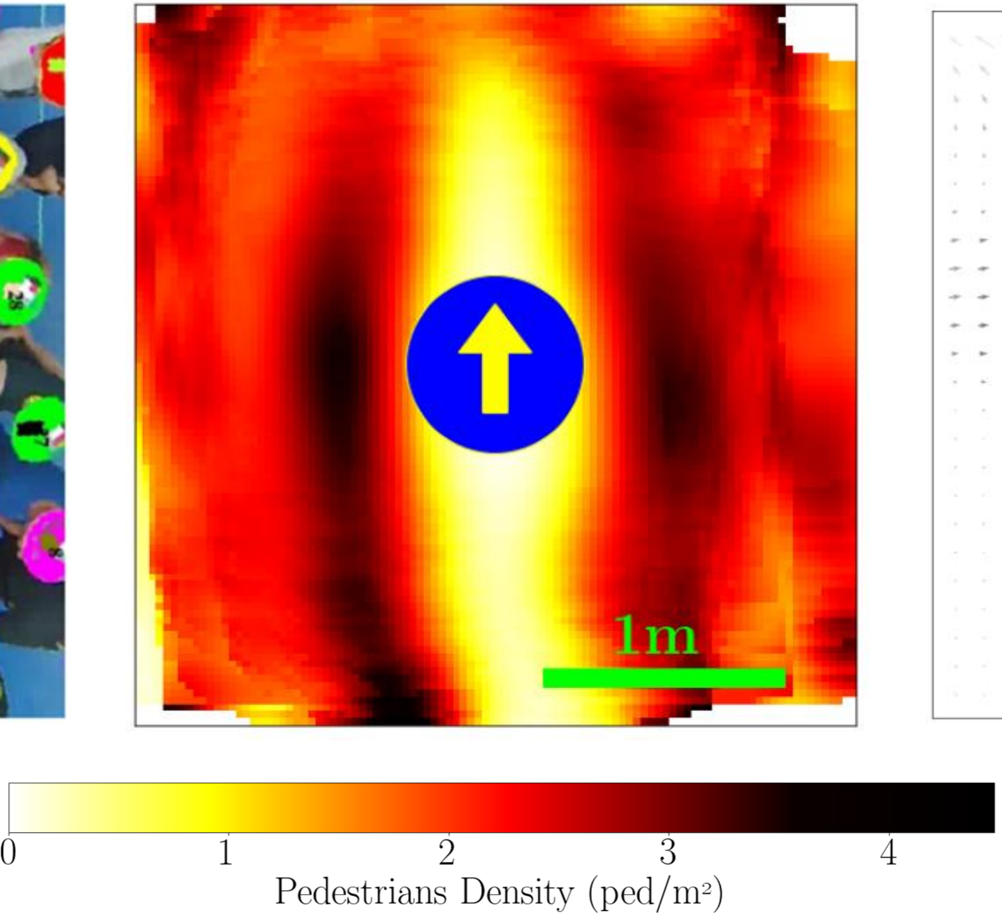


Symmetric density profile

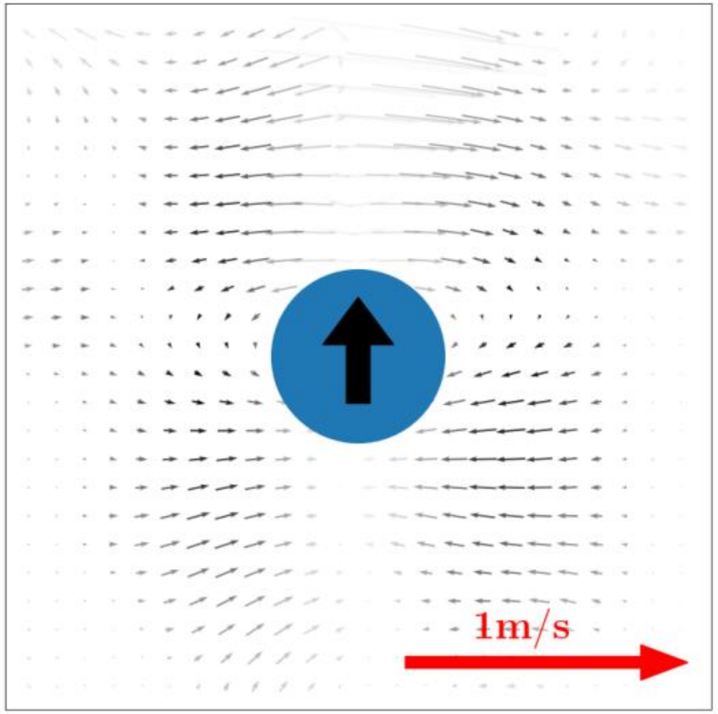
Intruder crossing a static crowd

Experiment

(Averaged) pedestrian density



Velocity field



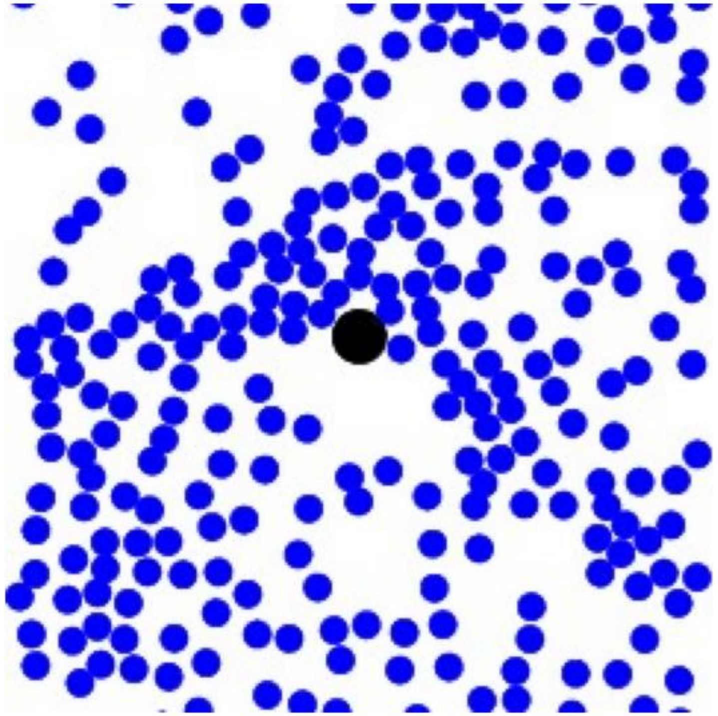
Symmetric density profile

Transverse displacement

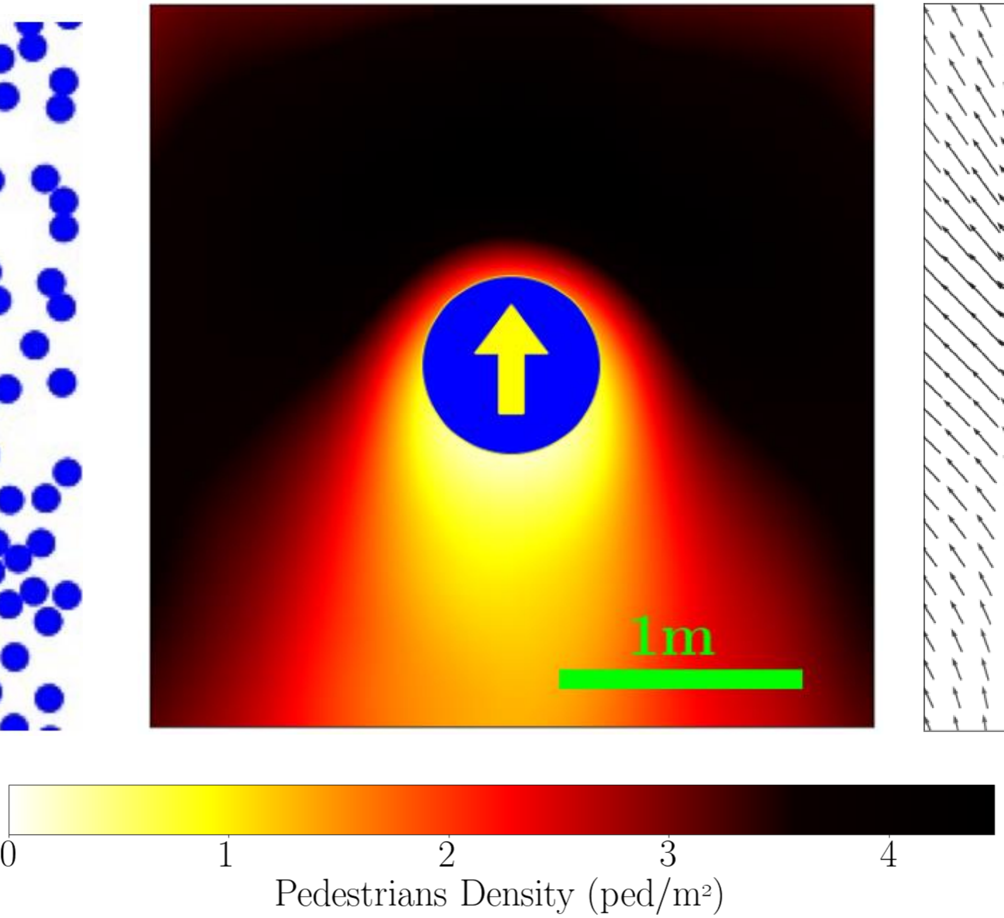
Granular matter

Based on [Seguin et al. (2009)]

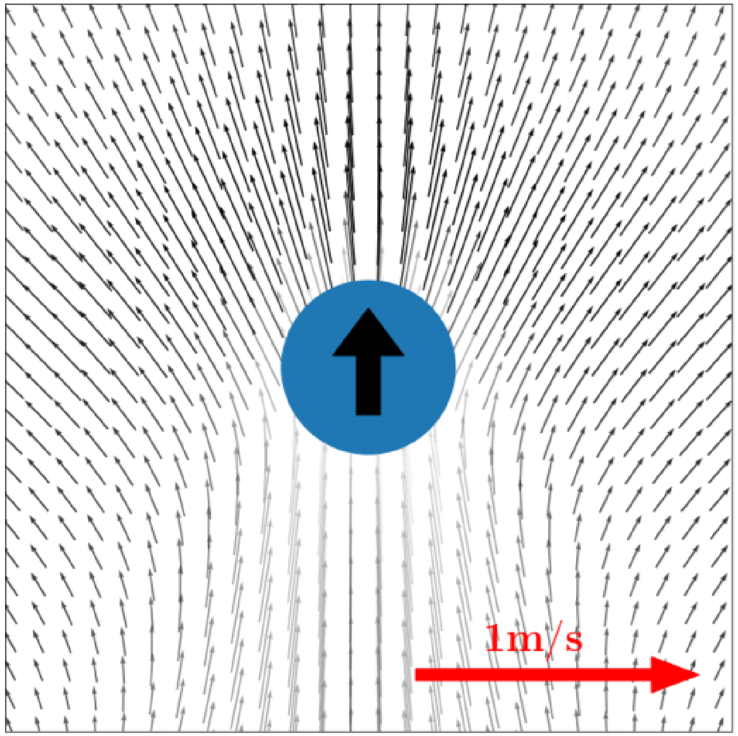
Simulation: monolayer of vibrated discs



(Averaged) pedestrian density



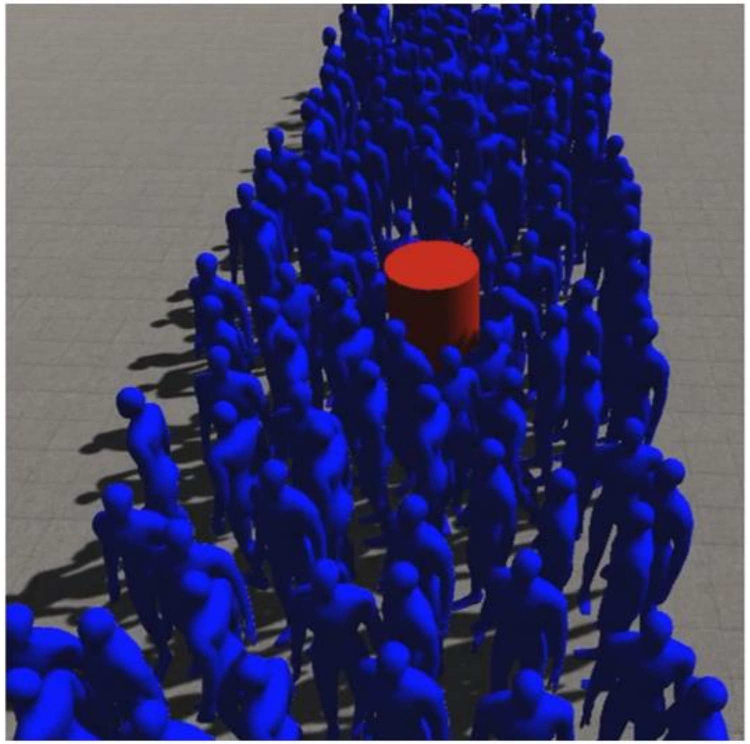
Velocity field



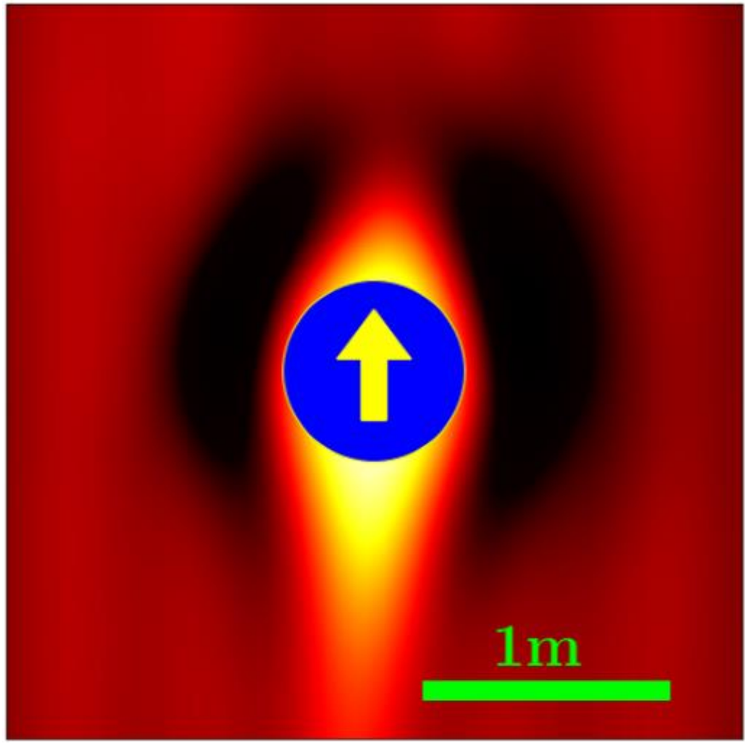
Time To Collision models

Based on [Echeverria-Huarte, Nicolas(2023)] and [Karamouzas et al. (2017)]

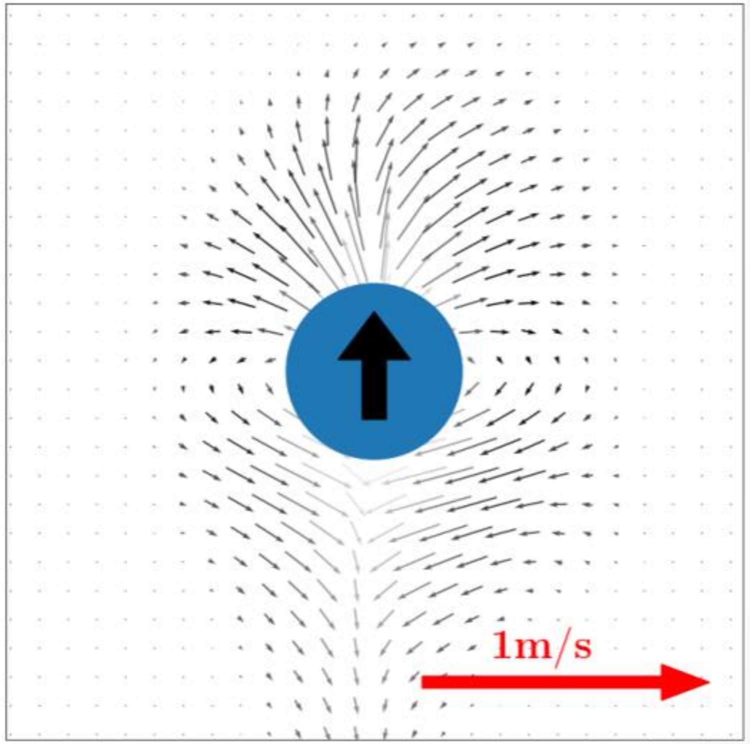
Simulation

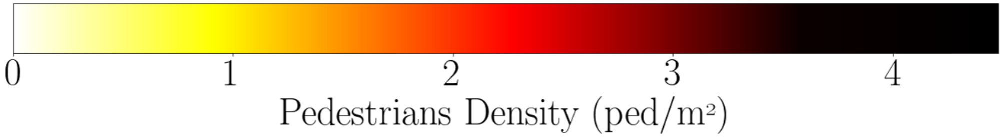


(Averaged) pedestrian density



Velocity field





To sum up…

- “Mechanical” models (granular and social forces) drastically fail to reproduce the qualitative features of the experiment.

To sum up…

- “Mechanical” models (granular and social forces) drastically fail to reproduce the qualitative features of the experiment.
- More modern TTC models struggle to do so.

To sum up...

- "Mechanical" models (granular and social forces) drastically fail to reproduce the qualitative features of the experiment.
- More modern TTC models struggle to do so.
- Experimental results are quite intuitive.

To sum up…

- “Mechanical” models (granular and social forces) drastically fail to reproduce the qualitative features of the experiment.
- More modern TTC models struggle to do so.
- Experimental results are quite intuitive.

↓ Change of paradigm

- Long-term anticipation → Competitive optimisation → Game theory

To sum up…

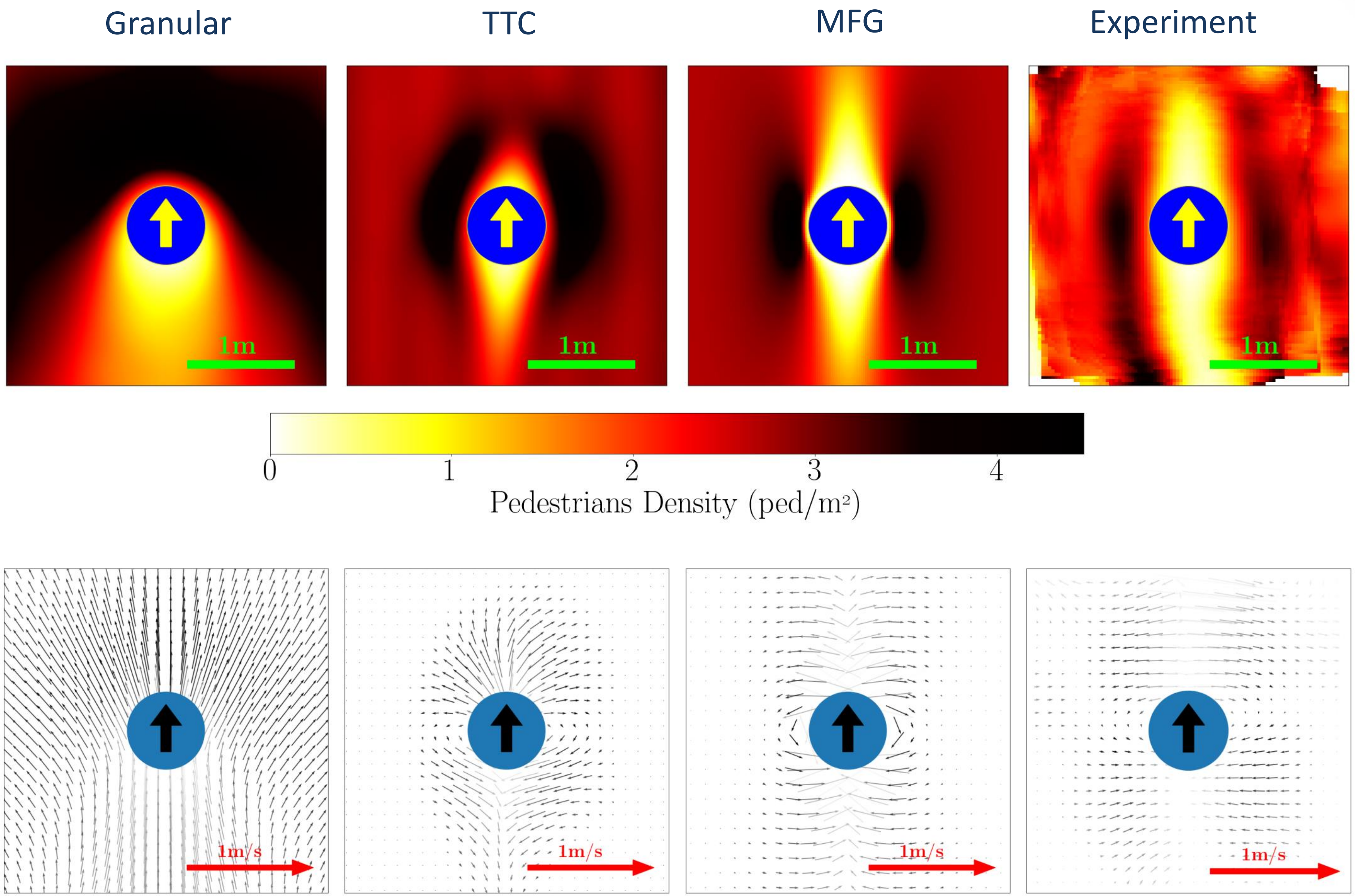
- "Mechanical" models (granular and social forces) drastically fail to reproduce the qualitative features of the experiment.
- More modern TTC models struggle to do so.
- Experimental results are quite intuitive.

Change of paradigm

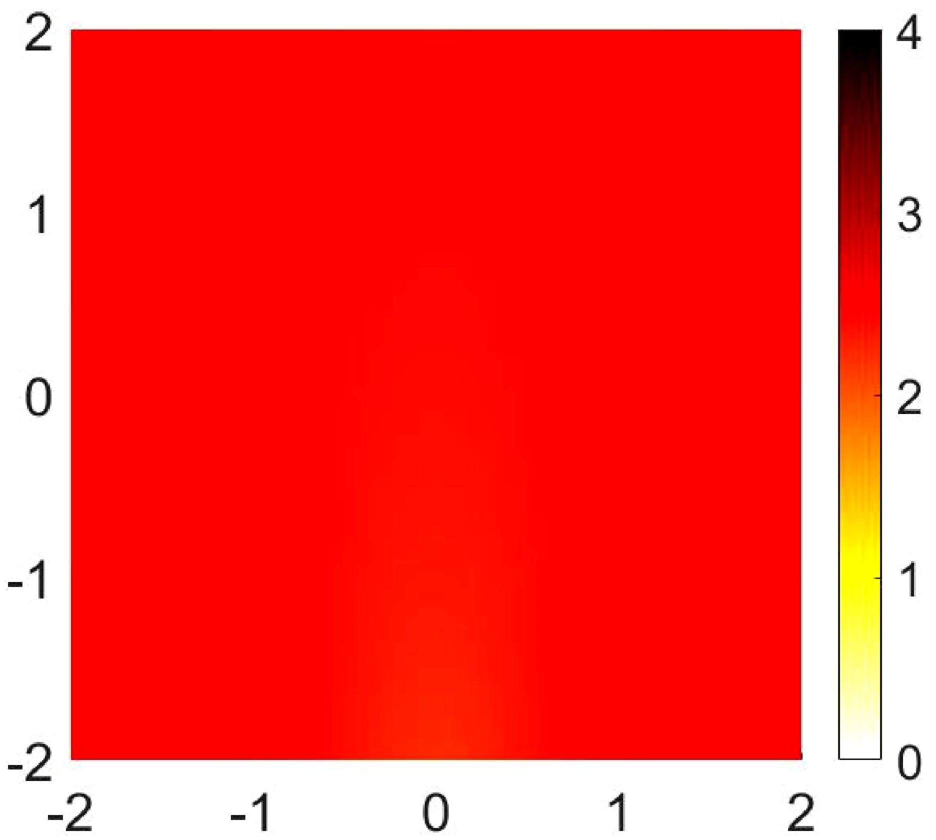
- Long-term anticipation $\rightarrow$ Competitive optimisation $\rightarrow$ Game theory
- Dense crowd $\rightarrow$ Many-body problem $\rightarrow$ Mean-field

Comparing the different approaches

[Bonnemain et al. (2023)]



Another example in a non-controlled environment



Game theory

Def: Mathematical framework to study strategic optimization.

Game theory

Def: Mathematical framework to study strategic optimization.

Hawk-Dove paradigm

	H	D
H	(¼ ; ¼)	(1 ; 0)
D	(0 ; 1)	(½ ; ½)

Mean Field Games

- Subdiscipline of Game Theory: problems of optimization with interacting agents in the large N limit.

Mean Field Games

- Subdiscipline of Game Theory: problems of optimization with interacting agents in the large N limit.

- Relatively recent: seminal papers published in 2006.

[Lasry, Lions (2006)] *[Huang, Malhamé, Caines (2006)]*

Mean Field Games

- Subdiscipline of Game Theory: problems of optimization with interacting agents in the large N limit.

- Relatively recent: seminal papers published in 2006.

  [Lasry, Lions (2006)] *[Huang, Malhamé, Caines (2006)]*

- Wide litterature: mathematics, engineering sciences, economics, sociology ...

  [Lachapelle, Wolfram (2011)] *[Guéant et al. (2012)]*
  [Gomes, Saude (2014)] *[Laguzet, Turinici (2015)]*
  [Achdou et al. (2016)] *[Cardaliaguet , Lehalle (2017)]*
  [Bremaud, Ullmo (2022)] *[Bonnemain et al. (2023)]*

MFG paradigm: population dynamics

[Guéant, Lasry, Lions (2011)]

- N agents $i = 1, 2, \cdots, N \quad (N \gg 1)$

MFG paradigm: population dynamics

[Guéant, Lasry, Lions (2011)]

- N agents $i = 1, 2, \cdots, N \quad (N \gg 1)$
- state of agent $i \longrightarrow$ real vector $\mathbf{X}^i$ (position, capital, beliefs...)

$$m(\mathbf{x}, t) \equiv \frac{1}{N} \sum_{1}^{N} \delta(\mathbf{x} - \mathbf{X}_t^i) \qquad \text{density of agents}$$

MFG paradigm: population dynamics

[Guéant, Lasry, Lions (2011)]

- N agents $i = 1, 2, \cdots, N \quad (N \gg 1)$
- state of agent $i \longrightarrow$ real vector $\mathbf{X}^i$ (position, capital, beliefs...)

$$m(\mathbf{x}, t) \equiv \frac{1}{N} \sum_{1}^{N} \delta(\mathbf{x} - \mathbf{X}_t^i) \qquad \text{density of agents}$$

- agent's dynamic

$$\mathrm{d}\mathbf{X}_t^i = \mathbf{a}_t^i \mathrm{d}t + \sigma \mathrm{d}\mathbf{w}_t^i$$

$\mathrm{d}\mathbf{w}_t^i \equiv$ white noise
drift $\mathbf{a}_t^i \equiv$ control parameter

MFG paradigm: population dynamics

[Guéant, Lasry, Lions (2011)]

- N agents $i = 1, 2, \cdots, N \quad (N \gg 1)$
- state of agent $i \longrightarrow$ real vector $\mathbf{X}^i$ (position, capital, beliefs...)

$$m(\mathbf{x}, t) \equiv \frac{1}{N} \sum_{1}^{N} \delta(\mathbf{x} - \mathbf{X}_t^i) \qquad \text{density of agents}$$

- agent's dynamic

$$\mathrm{d}\mathbf{X}_t^i = \mathbf{a}_t^i \mathrm{d}t + \sigma \mathrm{d}\mathbf{w}_t^i$$

$\mathrm{d}\mathbf{w}_t^i \equiv$ white noise
drift $\mathbf{a}_t^i \equiv$ control parameter

- agent tries to optimize (by the proper choice of $\mathbf{a}_t^i$) the cost function

$$\mathbb{E}\left[\int_t^T \Big(\underbrace{\frac{\mu}{2}(\mathbf{a}_s^i)^2 - V[m](\mathbf{X}_s^i, s)}_{\text{running cost}}\Big)\mathrm{d}s + \underbrace{c_T(\mathbf{X}_T^i)}_{\text{terminal cost}}\right]$$

MFG paradigm: population dynamics

[Guéant, Lasry, Lions (2011)]

- N agents $i = 1, 2, \cdots, N \quad (N \gg 1)$
- state of agent $i \longrightarrow$ real vector $\mathbf{X}^i$ (position, capital, beliefs...)

$$m(\mathbf{x}, t) \equiv \frac{1}{N} \sum_{1}^{N} \delta(\mathbf{x} - \mathbf{X}_t^i) \qquad \text{density of agents}$$

- agent's dynamic

$$d\mathbf{X}_t^i = \mathbf{a}_t^i dt + \sigma d\mathbf{w}_t^i$$

$d\mathbf{w}_t^i \equiv$ white noise
drift $\mathbf{a}_t^i \equiv$ control parameter

- agent tries to optimize (by the proper choice of $\mathbf{a}_t^i$) the cost function

$$\mathbb{E}\left[\int_t^T \underbrace{\left(\frac{\mu}{2}(\mathbf{a}_s^i)^2 - V[m](\mathbf{X}_s^i, s)\right)}_{\text{running cost}} ds + \underbrace{c_T(\mathbf{X}_T^i)}_{\text{terminal cost}}\right]$$

MFG paradigm: population dynamics

[Guéant, Lasry, Lions (2011)]

- N agents $i = 1, 2, \cdots, N \quad (N \gg 1)$
- state of agent $i \longrightarrow$ real vector $\mathbf{X}^i$ (position, capital, beliefs...)

$$m(\mathbf{x}, t) \equiv \frac{1}{N} \sum_{1}^{N} \delta(\mathbf{x} - \mathbf{X}_t^i) \qquad \text{density of agents}$$

- agent's dynamic

$$\mathrm{d}\mathbf{X}_t^i = \mathbf{a}_t^i \mathrm{d}t + \sigma \mathrm{d}\mathbf{w}_t^i$$

$\mathrm{d}\mathbf{w}_t^i \equiv$ white noise
drift $\mathbf{a}_t^i \equiv$ control parameter

- agent tries to optimize (by the proper choice of $\mathbf{a}_t^i$) the cost function

$$\mathbb{E}\left[\int_t^T \underbrace{\left(\frac{\mu}{2}(\mathbf{a}_s^i)^2 - V[m](\mathbf{X}_s^i, s)\right)}_{\text{running cost}} \mathrm{d}s + \underbrace{c_T(\mathbf{X}_T^i)}_{\text{terminal cost}}\right]$$

MFG paradigm: population dynamics

[Guéant, Lasry, Lions (2011)]

- N agents $i = 1, 2, \cdots, N \quad (N \gg 1)$
- state of agent $i \longrightarrow$ real vector $\mathbf{X}^i$ (position, capital, beliefs...)

$$m(\mathbf{x}, t) \equiv \frac{1}{N} \sum_{1}^{N} \delta(\mathbf{x} - \mathbf{X}_t^i) \qquad \text{density of agents}$$

- agent's dynamic

$$\mathrm{d}\mathbf{X}_t^i = \mathbf{a}_t^i \mathrm{d}t + \sigma \mathrm{d}\mathbf{w}_t^i$$

$\mathrm{d}\mathbf{w}_t^i \equiv$ white noise
drift $\mathbf{a}_t^i \equiv$ control parameter

- agent tries to optimize (by the proper choice of $\mathbf{a}_t^i$) the cost function

$$\mathbb{E}\left[\int_t^T \left(\underbrace{\frac{\mu}{2}(\mathbf{a}_s^i)^2 - V[m](\mathbf{X}_s^i, s)}_{\text{running cost}}\right)\mathrm{d}s + \underbrace{c_T(\mathbf{X}_T^i)}_{\text{terminal cost}}\right]$$

MFG paradigm: population dynamics

[Guéant, Lasry, Lions (2011)]

- N agents $i = 1, 2, \cdots, N \quad (N \gg 1)$
- state of agent $i \longrightarrow$ real vector $\mathbf{X}^i$ (position, capital, beliefs...)

$$m(\mathbf{x}, t) \equiv \frac{1}{N} \sum_{1}^{N} \delta(\mathbf{x} - \mathbf{X}_t^i) \qquad \text{density of agents}$$

- agent's dynamic

$$d\mathbf{X}_t^i = \mathbf{a}_t^i dt + \sigma d\mathbf{w}_t^i$$

$d\mathbf{w}_t^i \equiv$ white noise
drift $\mathbf{a}_t^i \equiv$ control parameter

- agent tries to optimize (by the proper choice of $\mathbf{a}_t^i$) the cost function

$$\mathbb{E}\left[\int_t^T \left(\underbrace{\frac{\mu}{2}(\mathbf{a}_s^i)^2 - g\,m(\mathbf{X}_s^i, s) - U_0(\mathbf{X}_s^i - \mathbf{v}t)}_{\text{running cost}}\right) ds + \underbrace{c_T(\mathbf{X}_T^i)}_{\text{terminal cost} = 0}\right]$$

Mean Field Games equations

- Optimization: linear programming leads to a *(backward)* Hamilton-Jacobi-Bellman equation for the value function $u(\mathbf{x}, t)$

$$\begin{cases} \partial_t u + \frac{1}{2\mu}(\nabla_{\mathbf{x}} u)^2 + \frac{\sigma^2}{2}\Delta_{\mathbf{x}} u = V[m](x,t) \\ u(x, t{=}T) = c_T(x) \end{cases} \qquad \text{(HJB)} \ .$$

Mean Field Games equations

- Optimization: linear programming leads to a *(backward)* Hamilton-Jacobi-Bellman equation for the value function $u(\mathbf{x}, t)$

$$\begin{cases} \partial_t u + \frac{1}{2\mu}\left(\nabla_{\mathbf{x}} u\right)^2 + \frac{\sigma^2}{2}\Delta_{\mathbf{x}} u = V[m](x,t) \\ u(x, t{=}T) = c_T(x) \end{cases} \qquad \text{(HJB)} .$$

- Langevin dynamic $d\mathbf{X}_t^i = \mathbf{a}_t^i dt + \sigma d\mathbf{w}_t^i$ leads to a *(forward)* diffusion equation for the density $m(x,t)$

$$\begin{cases} \partial_t m - \nabla_{\mathbf{x}}\left[m \frac{\nabla_{\mathbf{x}} u}{\mu}\right] - \frac{\sigma^2}{2}\Delta_{\mathbf{x}} m = 0 \\ m(x, t{=}0) = m_0(x) \end{cases} \qquad \text{(Kolmogorov)} .$$

Mean Field Games equations

- Optimization: linear programming leads to a *(backward)* Hamilton-Jacobi-Bellman equation for the value function $u(\mathbf{x}, t)$

$$\begin{cases} \partial_t u + \frac{1}{2\mu}(\nabla_{\mathbf{x}} u)^2 + \frac{\sigma^2}{2}\Delta_{\mathbf{x}} u = V[m](x,t) \\ u(x, t{=}T) = c_T(x) \end{cases} \quad \text{(HJB)} .$$

- Langevin dynamic $\mathrm{d}\mathbf{X}_t^i = \mathbf{a}_t^i \mathrm{d}t + \sigma \mathrm{d}\mathbf{w}_t^i$ leads to a *(forward)* diffusion equation for the density $m(x,t)$

$$\begin{cases} \partial_t m - \nabla_{\mathbf{x}}\left[m \frac{\nabla_{\mathbf{x}} u}{\mu}\right] - \frac{\sigma^2}{2}\Delta_{\mathbf{x}} m = 0 \\ m(x, t{=}0) = m_0(x) \end{cases} \quad \text{(Kolmogorov)} .$$

optimal control $\mathbf{a}$

Mean Field Game = coupling between a (collective) stochastic motion and an (individual) optimization problem through a mean field

Ergodic state

Theorem: *[Cardaliaguet, Lasry, Lions, Poretta (2013)]*

- V[m](x,t) has no explicit time dependence
- Long optimization time: $T \to \infty$
- System is confined
- ... + other conditions ...

$\Rightarrow$ for $0 \ll t \ll T$
$$\left| \begin{array}{l} m(\mathbf{x},t) \simeq m_{\mathrm{e}}(\mathbf{x}) \\ u(\mathbf{x},t) \simeq u_{\mathrm{e}}(\mathbf{x}) + \lambda t \end{array} \right.$$

Ergodic state

Theorem: *[Cardaliaguet, Lasry, Lions, Poretta (2013)]*

- V[m](x,t) has no explicit time dependence
- Long optimization time: $T \to \infty$
- System is confined
- ... + other conditions ...

$\Rightarrow$ for $0 \ll t \ll T$ $\quad \begin{cases} m(\mathbf{x},t) \simeq m_e(\mathbf{x}) \\ u(\mathbf{x},t) \simeq u_e(\mathbf{x}) + \lambda t \end{cases}$

(m_e, u_e, λ) such that

$$\begin{cases} \lambda - \frac{1}{2\mu}(\nabla_{\mathbf{x}} u_e)^2 + \frac{\sigma^2}{2}\Delta_{\mathbf{x}} u_e = V[m_e](\mathbf{x}) \\ \nabla_{\mathbf{x}}(m_e(\nabla_{\mathbf{x}} u_e)) - \frac{\sigma^2}{2}\Delta_{\mathbf{x}} m_e = 0 \end{cases}$$

Ergodic state

Theorem: *[Cardaliaguet, Lasry, Lions, Poretta (2013)]*

- V[m](x,t) has no explicit time dependence
- Long optimization time: $T \to \infty$
- System is confined
- … + other conditions …

$V[m](\mathbf{x},t) = gm(\mathbf{x},t) + U_0(\mathbf{x}-\mathbf{v}t)$

Is this relevant?

⇨ for $0 \ll t \ll T$ $\left| \begin{array}{l} m(\mathbf{x},t) \simeq m_e(\mathbf{x}) \\ u(\mathbf{x},t) \simeq u_e(\mathbf{x}) + \lambda t \end{array} \right.$

(m_e, u_e, λ) such that $\begin{cases} \lambda - \dfrac{1}{2\mu}(\nabla_{\mathbf{x}} u_e)^2 + \dfrac{\sigma^2}{2}\Delta_{\mathbf{x}} u_e = V[m_e](\mathbf{x}) \\ \nabla_{\mathbf{x}}(m_e(\nabla_{\mathbf{x}} u_e)) - \dfrac{\sigma^2}{2}\Delta_{\mathbf{x}} m_e = 0 \end{cases}$

Ergodic state

Theorem:

[Cardaliaguet, Lasry, Lions, Poretta (2013)]

- V[m](x,t) has no explicit time dependence
- Long optimization time: $T \to \infty$
- System is confined
- ... + other conditions ...

$V[m](\mathbf{x}, t) = g m(\mathbf{x}, t) + U_0(\mathbf{x} - \mathbf{v}t)$

Is this relevant?

$\mathbf{y} \equiv \mathbf{x} - \mathbf{v}t$ ⇒

for $0 \ll t \ll T$ $\begin{cases} m(\mathbf{y}, t) \simeq m_\mathrm{e}(\mathbf{y}) \\ u(\mathbf{y}, t) \simeq u_\mathrm{e}(\mathbf{y}) + \lambda t \end{cases}$

$(m_\mathrm{e}, u_\mathrm{e}, \lambda)$ such that

$$\begin{cases} \lambda - \mathbf{v} \cdot \nabla_\mathbf{y} - \frac{1}{2\mu} (\nabla_\mathbf{y} u_\mathrm{e})^2 + \frac{\sigma^2}{2} \Delta_\mathbf{y} u_\mathrm{e} = V[m_\mathrm{e}](\mathbf{x}) \\ \nabla_\mathbf{y}(m_\mathrm{e}(\nabla_\mathbf{y} u_e) + \mathbf{v}) - \frac{\sigma^2}{2} \Delta_\mathbf{y} m_\mathrm{e} = 0 \end{cases}$$

Ergodic state

Theorem:

[Cardaliaguet, Lasry, Lions, Poretta (2013)]

- V[m](x,t) has no explicit time dependence
- Long optimization time: $T \to \infty$
- System is confined
- … + other conditions …

$V[m](\mathbf{x},t) = gm(\mathbf{x},t) + U_0(\mathbf{x}-\mathbf{v}t)$

Is this relevant?

$\mathbf{y} \equiv \mathbf{x} - \mathbf{v}t$ ⇒

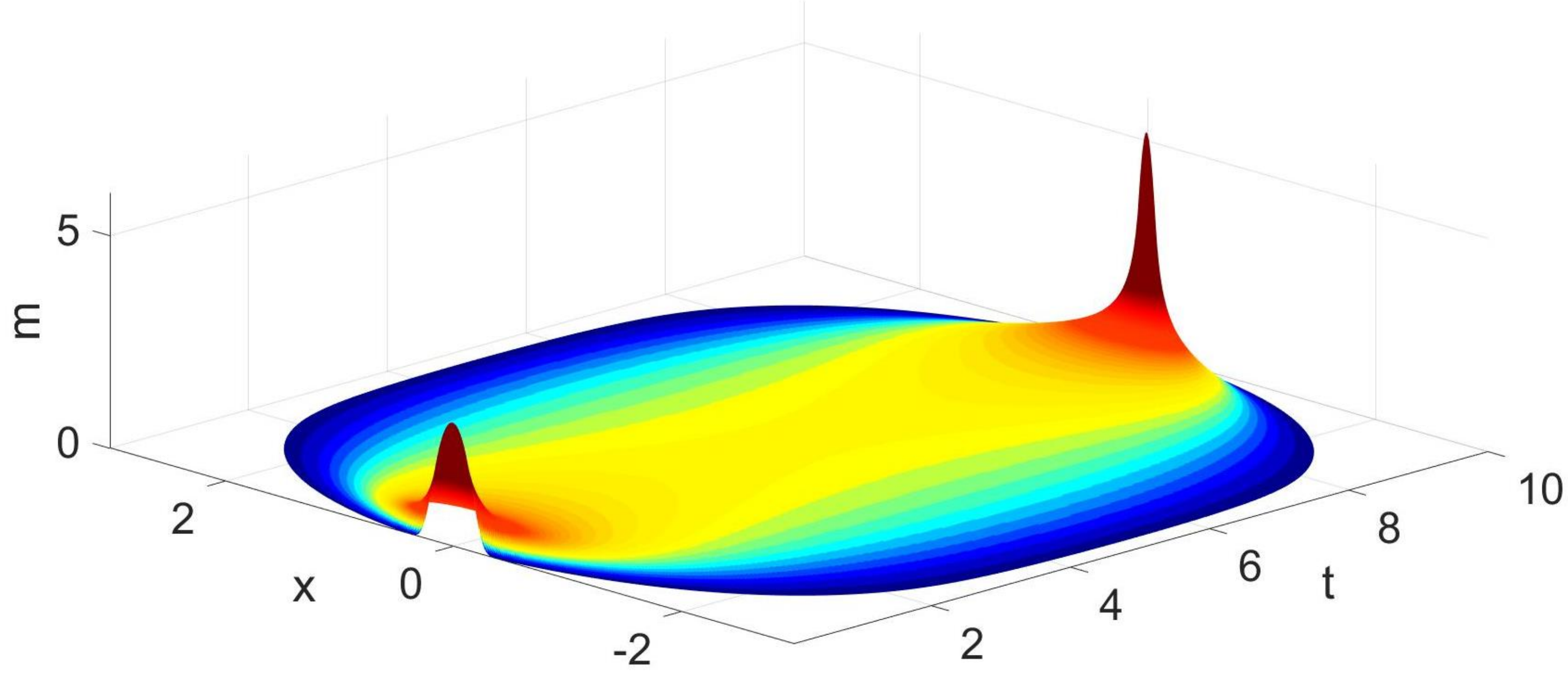
$$\text{for } 0 \ll t \ll T \quad \left| \begin{array}{l} m(\mathbf{y},t) \simeq m_{\mathrm{e}}(\mathbf{y}) \\ u(\mathbf{y},t) \simeq u_{\mathrm{e}}(\mathbf{y}) + \lambda t \end{array} \right.$$

$(m_{\mathrm{e}}, u_{\mathrm{e}}, \lambda)$ such that

$$\begin{cases} \lambda - \mathbf{v}\cdot\nabla_{\mathbf{y}} - \frac{1}{2\mu}(\nabla_{\mathbf{y}} u_{\mathrm{e}})^2 + \frac{\sigma^2}{2}\Delta_{\mathbf{y}} u_{\mathrm{e}} = V[m_{\mathrm{e}}](\mathbf{x}) \\ \nabla_{\mathbf{y}}(m_{\mathrm{e}}(\nabla_{\mathbf{y}} u_e) + \mathbf{v}) - \frac{\sigma^2}{2}\Delta_{\mathbf{y}} m_{\mathrm{e}} = 0 \end{cases}$$

$= -gm_0$

A simple example



"Search party" toy model: $V[m](x,t) = \underbrace{g}_{<0}m(x,t) + \underbrace{U_0(x)}_{\propto -x^2}$

Non-linear Schrödinger representation

[Bonnemain, Gobron, Ullmo Phys.Lett. A (2020) ; SciPost (2020) ; J. Math. Phys. (2021)]

Introduce two new variables $\Phi(\mathbf{x},t)$, $\Gamma(\mathbf{x},t)$ defined by:

$$u(\mathbf{x},t) = -\mu\sigma^2 \log\left(\Phi(\mathbf{x},t)\right) \qquad m(\mathbf{x},t) = \Gamma(\mathbf{x},t)\Phi(\mathbf{x},t)$$

$$\Rightarrow \begin{cases} \mu\sigma^2\partial_t\Gamma = \dfrac{\mu\sigma^4}{2}\Delta_{\mathbf{x}}\Gamma + U_0(\mathbf{x})\Gamma + g\,m\Gamma \\ -\mu\sigma^2\partial_t\Phi = \dfrac{\mu\sigma^4}{2}\Delta_{\mathbf{x}}\Phi + U_0(\mathbf{x})\Phi + g\,m\Phi \end{cases}$$

$m = \Gamma\Phi$

Non-linear Schrödinger representation

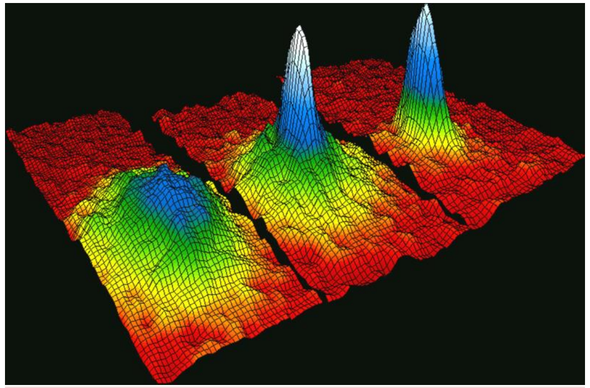
[Bonnemain, Gobron, Ullmo Phys.Lett. A (2020) ; SciPost (2020) ; J. Math. Phys. (2021)]

Introduce two new variables $\Phi(\mathbf{x},t)$, $\Gamma(\mathbf{x},t)$ defined by:

$$u(\mathbf{x},t) = -\mu\sigma^2 \log\left(\Phi(\mathbf{x},t)\right) \qquad m(\mathbf{x},t) = \Gamma(\mathbf{x},t)\Phi(\mathbf{x},t)$$

$$\Rightarrow \begin{cases} \mu\sigma^2 \partial_t \Gamma = \dfrac{\mu\sigma^4}{2}\Delta_{\mathbf{x}}\Gamma + U_0(\mathbf{x})\Gamma + g\, m\Gamma \\ -\mu\sigma^2 \partial_t \Phi = \dfrac{\mu\sigma^4}{2}\Delta_{\mathbf{x}}\Phi + U_0(\mathbf{x})\Phi + g\, m\Phi \end{cases}$$

$m = \Gamma\Phi$



Rubidium atoms (170 nK)

$$i\hbar\partial_t \Psi = -\frac{\hbar^2}{2\mu}\Delta_{\mathbf{x}}\Psi + U_0(\mathbf{x})\Psi + g|\Psi|^2\Psi$$

Non-Linear Schrödinger

$$(\Psi, \Psi^*, \hbar) \to (\Phi, \Gamma, i\mu\sigma^2)$$

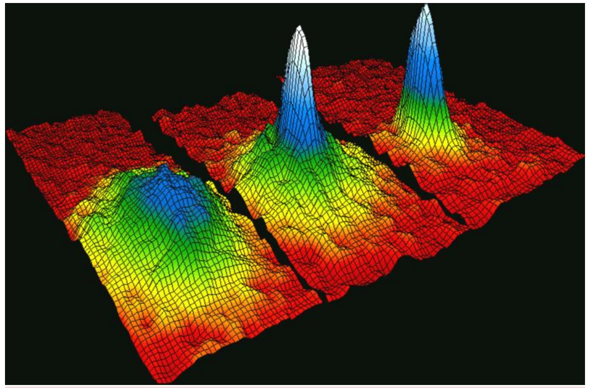
Non-linear Schrödinger representation

[Bonnemain, Gobron, Ullmo Phys.Lett. A (2020) ; SciPost (2020) ; J. Math. Phys. (2021)]

Introduce two new variables $\Phi_e(\mathbf{x},t)$, $\Gamma_e(\mathbf{x},t)$ defined by:

$$u_e(\mathbf{x},t) = -\mu\sigma^2 \log\left(\Phi_e(\mathbf{x},t)\right) \qquad m_e(\mathbf{x},t) = \Gamma_e(\mathbf{x},t)\Phi_e(\mathbf{x},t)$$

$$\Rightarrow \begin{cases} \lambda\Gamma_e = \frac{\mu\sigma^4}{2}\Delta_{\mathbf{x}}\Gamma_e + U_0(\mathbf{x})\Gamma_e + g\, m_e\Gamma_e \\ \lambda\Phi_e = \frac{\mu\sigma^4}{2}\Delta_{\mathbf{x}}\Phi_e + U_0(\mathbf{x})\Phi_e + g\, m_e\Phi_e \end{cases}$$



Rubidium atoms (170 nK)

$$\lambda'\Psi = -\frac{\hbar^2}{2\mu}\Delta_{\mathbf{x}}\Psi + U_0(\mathbf{x})\Psi + g|\Psi|^2\Psi$$

Non-Linear Schrödinger

$(\Psi, \Psi^*, \hbar) \to (\Phi, \Gamma, i\mu\sigma^2)$

Quantum mechanical formalism

[Ullmo, Swiecicki, Gobron (2019)]

- Operators: $\hat{X} \equiv \mathbf{x}$, $\quad \hat{\Pi} \equiv \mu\sigma^2\nabla_{\mathbf{x}}$, $\quad \hat{O} \equiv f(\hat{\Pi}, \hat{X})$

Quantum mechanical formalism

[Ullmo, Swiecicki, Gobron (2019)]

- Operators: $\hat{X} \equiv \mathbf{x}$, $\quad \hat{\Pi} \equiv \mu\sigma^2\nabla_{\mathbf{x}}$, $\quad \hat{O} \equiv f(\hat{\Pi}, \hat{X})$

- Average: $\langle\hat{O}\rangle(t) \equiv \int \mathrm{d}\mathbf{x}\ \Gamma(\mathbf{x}, t)\hat{O}\Phi(\mathbf{x}, t)$, $\quad m = \Gamma\Phi$

$$\text{e.g.} \qquad \langle\hat{N}\rangle \equiv \int \mathrm{d}\mathbf{x}\ m(\mathbf{x}, t) \ , \qquad \langle\hat{X}\rangle \equiv \int \mathrm{d}\mathbf{x}\ m(\mathbf{x}, t)\mathbf{x}$$

Quantum mechanical formalism

[Ullmo, Swiecicki, Gobron (2019)]

- Operators: $\hat{X} \equiv \mathbf{x}\ , \qquad \hat{\Pi} \equiv \mu\sigma^2 \nabla_{\mathbf{x}}\ , \qquad \hat{O} \equiv f(\hat{\Pi}, \hat{X})$
- Average: $\langle \hat{O} \rangle(t) \equiv \int \mathrm{d}\mathbf{x}\ \Gamma(\mathbf{x}, t) \hat{O} \Phi(\mathbf{x}, t)\ , \qquad m = \Gamma\Phi$
- Hamiltonian: $\hat{H} \equiv -\left(\frac{\hat{\Pi}}{2\mu} + g\Phi\Gamma + U_0 \right)$

$$\begin{cases} -\mu\sigma^2 \partial_t \Gamma = \hat{H}\Gamma \\ +\mu\sigma^2 \partial_t \Phi = \hat{H}\Phi \end{cases} \quad \Rightarrow \quad \mu\sigma^2 \frac{\mathrm{d}}{\mathrm{d}t} \langle \hat{O} \rangle = \langle \partial_t \hat{O} \rangle + \langle [\hat{O}, \hat{H}] \rangle$$

Quantum mechanical formalism

[Ullmo, Swiecicki, Gobron (2019)]

- Operators: $\hat{X} \equiv \mathbf{x}\ , \qquad \hat{\Pi} \equiv \mu\sigma^2 \nabla_{\mathbf{x}}\ , \qquad \hat{O} \equiv f(\hat{\Pi}, \hat{X})$

- Average: $\langle \hat{O} \rangle (t) \equiv \int \mathrm{d}\mathbf{x}\ \Gamma(\mathbf{x}, t) \hat{O} \Phi(\mathbf{x}, t)\ , \qquad m = \Gamma\Phi$

- Hamiltonian: $\hat{H} \equiv -\left(\frac{\hat{\Pi}}{2\mu} + g\Phi\Gamma + U_0 \right)$

$$\begin{cases} -\mu\sigma^2 \partial_t \Gamma = \hat{H}\Gamma \\ +\mu\sigma^2 \partial_t \Phi = \hat{H}\Phi \end{cases} \Rightarrow \quad \mu\sigma^2 \frac{\mathrm{d}}{\mathrm{d}t} \langle \hat{O} \rangle = \langle \partial_t \hat{O} \rangle + \langle [\hat{O}, \hat{H}] \rangle$$

e.g. $$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} \langle \hat{X} \rangle = \frac{\langle \hat{\Pi} \rangle}{\mu} \\ \frac{\mathrm{d}}{\mathrm{d}t} \langle \hat{\Pi} \rangle = \langle -\nabla_{\mathbf{x}} U_0(\hat{X}) \rangle \end{cases} \qquad \begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} \Sigma^2 = \frac{1}{\mu} \left(\langle \hat{X}\hat{\Pi} + \hat{\Pi}\hat{X} \rangle - 2\langle \hat{\Pi} \rangle \langle \hat{X} \rangle \right) \equiv \frac{\Lambda}{\mu} \\ \frac{\mathrm{d}}{\mathrm{d}t} \Lambda = -2\langle \hat{X} \nabla_{\mathbf{x}} U_0(\hat{X}) \rangle + 2\langle \hat{\Pi}^2 \rangle \end{cases}$$

$$\Sigma^2 \equiv \langle \hat{X}^2 \rangle - \langle \hat{X} \rangle^2$$

Action and variational approach

- Quadratic MFG are variational systems

$$S[\Gamma,\Phi] \equiv \int_0^T \mathrm{d}t \int_{\mathbb{R}} \mathrm{d}\mathbf{x} \left[\frac{\mu\sigma^2}{2}(\Gamma\partial_t\Phi - \Phi\partial_t\Gamma) - \frac{\mu\sigma^4}{2}\nabla\Gamma.\nabla\Phi + \left[U_0 + \frac{g}{2}\Gamma\Phi\right]\Gamma\Phi \right]$$

Action and variational approach

- Quadratic MFG are variational systems

$$S[\Gamma, \Phi] \equiv \int_0^T \mathrm{d}t \int_{\mathbb{R}} \mathrm{d}\mathbf{x} \left[\frac{\mu\sigma^2}{2} (\Gamma \partial_t \Phi - \Phi \partial_t \Gamma) - \frac{\mu\sigma^4}{2} \nabla\Gamma . \nabla\Phi + \left[U_0 + \frac{g}{2} \Gamma\Phi \right] \Gamma\Phi \right]$$

- Noether theorem: Energy is conserved

$$E = \int_{\mathbb{R}} \mathrm{d}\mathbf{x} \left[-\frac{\mu\sigma^4}{2} \nabla\Gamma . \nabla\Phi + \frac{g}{2} (\Gamma\Phi)^2 + U_0 \Gamma\Phi \right]$$

Action and variational approach

- Quadratic MFG are variational systems

$$S[\Gamma,\Phi] \equiv \int_0^T \mathrm{d}t \int_{\mathbb{R}} \mathrm{d}\mathbf{x} \left[\frac{\mu\sigma^2}{2}(\Gamma\partial_t\Phi - \Phi\partial_t\Gamma) - \frac{\mu\sigma^4}{2}\nabla\Gamma.\nabla\Phi + \left[U_0 + \frac{g}{2}\Gamma\Phi\right]\Gamma\Phi \right]$$

- Noether theorem: Energy is conserved

$$E = \int_{\mathbb{R}} \mathrm{d}\mathbf{x} \left[-\frac{\mu\sigma^4}{2}\nabla\Gamma.\nabla\Phi + \frac{g}{2}(\Gamma\Phi)^2 + U_0\Gamma\Phi \right]$$

E^{kin} E^{int} E^{pot}

Action and variational approach

- Quadratic MFG are variational systems

$$S[\Gamma,\Phi] \equiv \int_0^T \mathrm{d}t \int_{\mathbb{R}} \mathrm{d}\mathbf{x} \left[\frac{\mu\sigma^2}{2}(\Gamma\partial_t\Phi - \Phi\partial_t\Gamma) - \frac{\mu\sigma^4}{2}\nabla\Gamma.\nabla\Phi + \left[U_0 + \frac{g}{2}\Gamma\Phi\right]\Gamma\Phi \right]$$

- Noether theorem: Energy is conserved

$$E = \int_{\mathbb{R}} \mathrm{d}\mathbf{x} \left[-\frac{\mu\sigma^4}{2}\nabla\Gamma.\nabla\Phi + \frac{g}{2}(\Gamma\Phi)^2 + U_0\Gamma\Phi \right]$$

E^{kin} $\quad E^{\mathrm{int}}$ $\quad E^{\mathrm{pot}}$

$$\Rightarrow \quad \frac{E^{\mathrm{kin}}}{E^{\mathrm{int}}} \sim \frac{\xi}{L} \qquad \xi_{1\mathrm{D}} = \frac{\mu\sigma^4}{|g|} \qquad \xi_{2\mathrm{D}} = \sqrt{\frac{\mu\sigma^4}{|g|m_0}}$$

Solitons and integrability in (1+1)D

[Bonnemain, Gobron, Ullmo (2021)]

- In (1+1)D, if $U_0 = 0$: NLS and MFG are integrable.
- Lax connection

$$\partial_t U - \partial_x V + [U,V] = 0$$

$$U = \kappa_\epsilon \begin{pmatrix} \frac{\lambda}{2} & \Phi \\ \Gamma & -\frac{\lambda}{2} \end{pmatrix} \quad , \qquad V = \kappa_\epsilon \begin{pmatrix} \kappa_\epsilon \Phi\Gamma & -\partial_x \Phi \\ \partial_x \Gamma & -\kappa_\epsilon \Phi\Gamma \end{pmatrix} - \lambda U$$

- Poisson bracket and infinite hierearchy of conservation laws

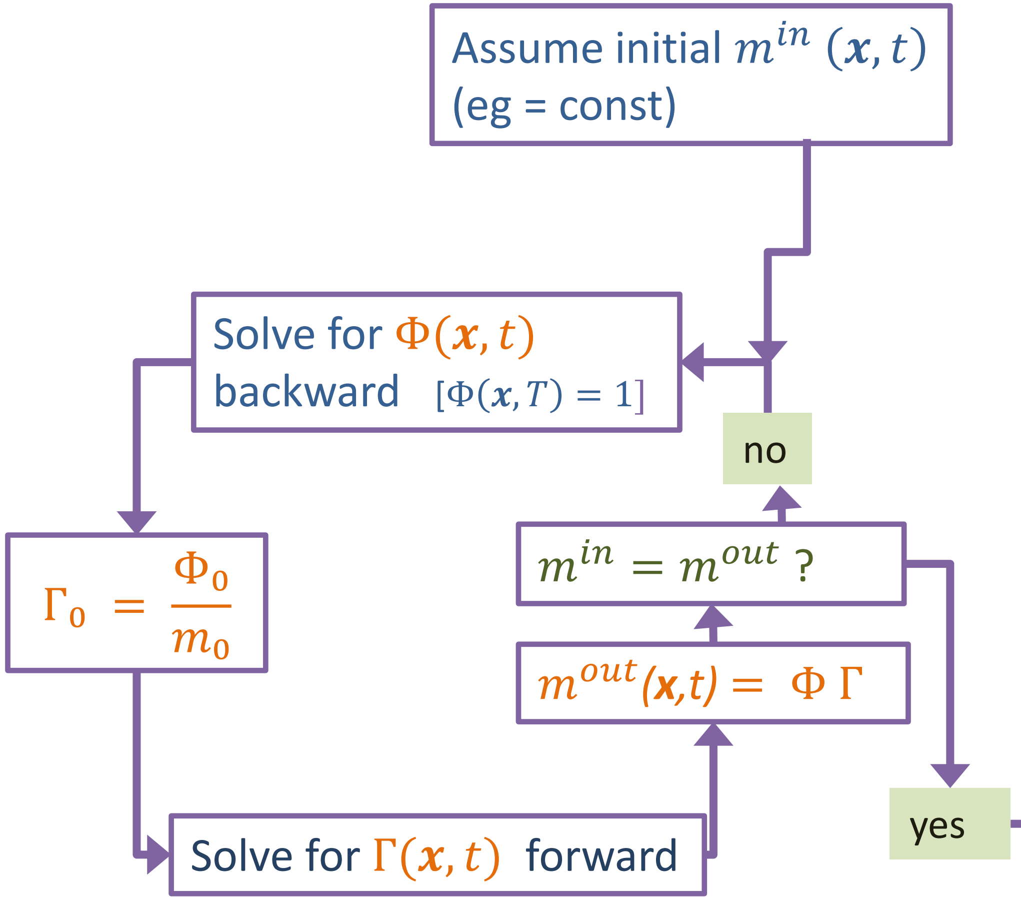
$$\{F,G\} = \int_{\mathbb{R}} \left(\frac{\delta F}{\delta \Gamma}\frac{\delta G}{\delta \Phi} - \frac{\delta F}{\delta \Phi}\frac{\delta G}{\delta \Gamma} \right) \mathrm{d}x \quad , \qquad \{Q_n, Q_m\} = 0$$

- Soliton solutions

$$\Psi(x,t) = 2b\mathrm{sech}\left[2b(x + 4at - x_0)\right] e^{\pm 2\left[ax + 2(a^2 - b^2)t + \phi_0\right]}$$

Back to our model of pedestrian dynamics

Numerical implementation



Propagation :

$$\begin{cases} -\mu\sigma^2\partial_t\Phi = \frac{\mu\sigma^4}{2}\Delta\Phi + (U_0 + gm^{\text{in}})\Phi \\ +\mu\sigma^2\partial_t\Gamma = \frac{\mu\sigma^4}{2}\Delta\Gamma + (U_0 + gm^{\text{in}})\Gamma \end{cases}$$

$[\Phi(T,\cdot) = 1]$ $\quad$ $[\Gamma(0,\cdot) = m_0(\cdot)/\Phi(0,\cdot)]$

Self consistent equation :

$$m^{\text{out}}(\mathbf{x}, t) \equiv \Gamma(\mathbf{x}, t)\Phi(\mathbf{x}, t) = m^{\text{in}}(\mathbf{x}, t)$$

Back to our model of pedestrian dynamics

Numerical implementation

Assume initial $m^{in}(\boldsymbol{x}, t)$ (eg = const)

Solve for $\Phi(\boldsymbol{x}, t)$ backward $[\Phi(\boldsymbol{x}, T) = 1]$

$\Gamma_0 = \frac{\Phi_0}{m_0}$

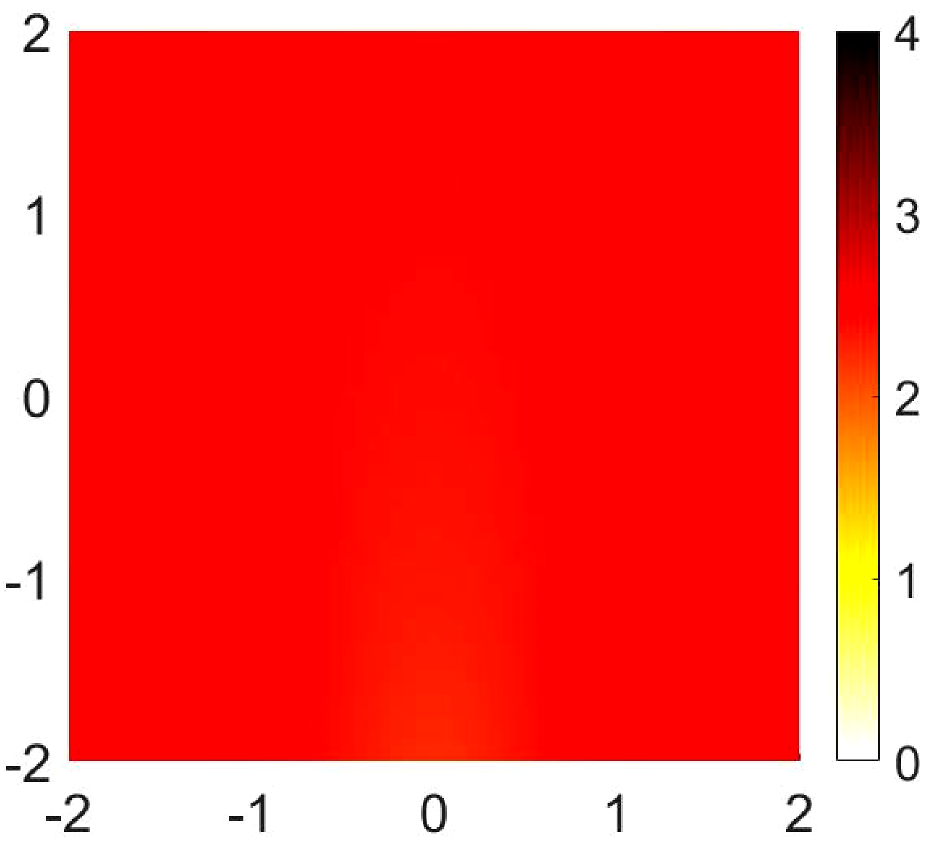
Solve for $\Gamma(\boldsymbol{x}, t)$ forward

$m^{out}(\boldsymbol{x},t) = \Phi\,\Gamma$

$m^{in} = m^{out}$?

no

yes



Ergodic vs time dependent

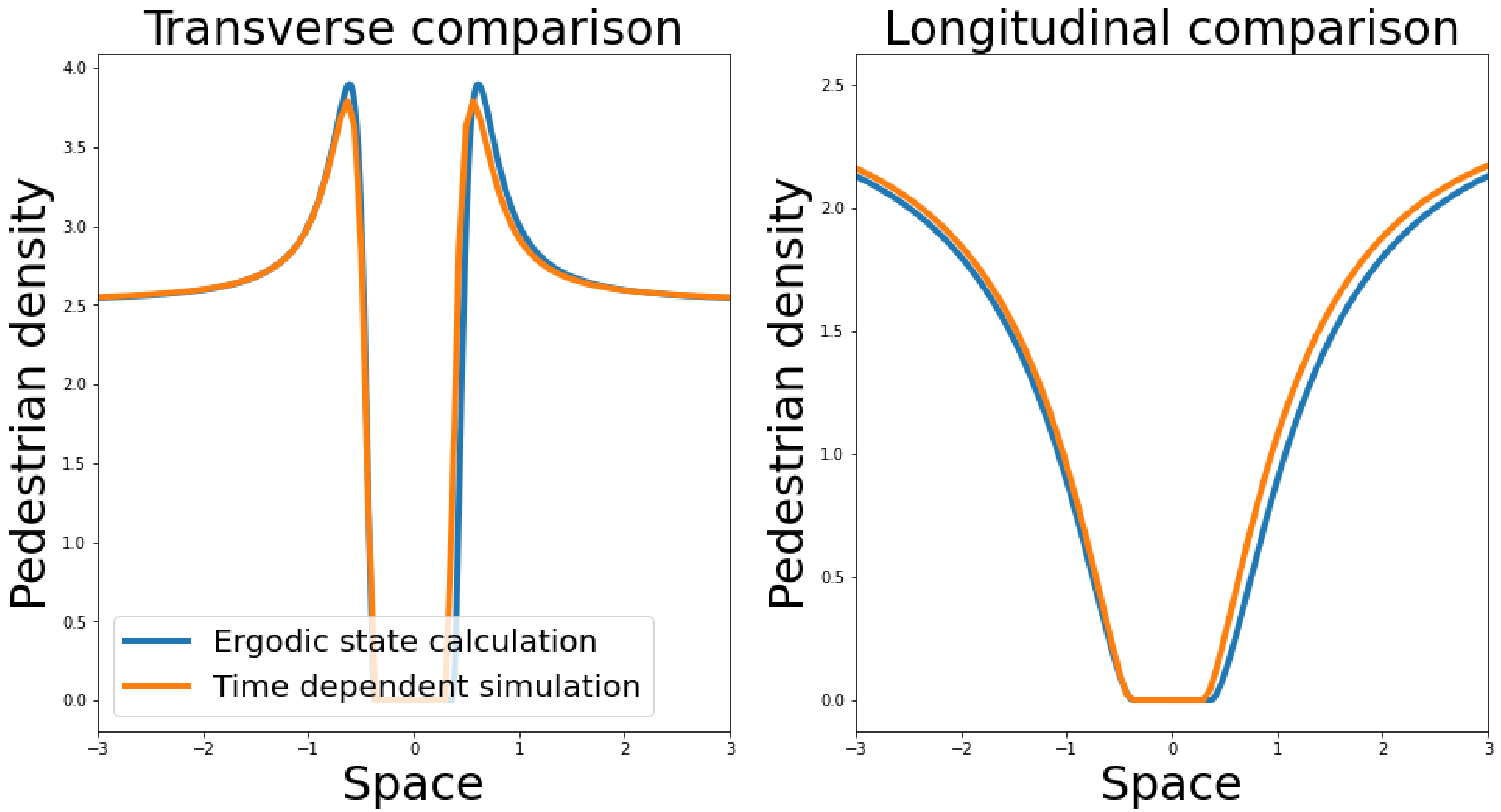
NB : exact symmetry for ergodic case

$$\begin{cases} -gm_0\Gamma_\mathrm{e}(\mathbf{y}) = \dfrac{\mu\sigma^4}{2}\Delta_\mathbf{y}\Gamma_\mathrm{e}(\mathbf{y}) + \mathbf{v}\cdot\nabla_\mathbf{y}\Gamma_\mathrm{e}(\mathbf{y}) + U_0(\mathbf{y})\Gamma_\mathrm{e}(\mathbf{y}) + g\,m_e(\mathbf{y})\Gamma_\mathrm{e}(\mathbf{y}) \\ -gm_0\Phi_\mathrm{e}(\mathbf{y}) = \dfrac{\mu\sigma^4}{2}\Delta_\mathbf{y}\Phi_\mathrm{e}(\mathbf{y}) - \mathbf{v}\cdot\nabla_\mathbf{y}\Phi_\mathrm{e}(\mathbf{y}) + U_0(\mathbf{y})\Phi_\mathrm{e}(\mathbf{y}) + g\,m_\mathrm{e}(\mathbf{y})\Phi_\mathrm{e}(\mathbf{y}) \end{cases}$$

Ergodic vs time dependent

NB : exact symmetry for ergodic case

$$\begin{cases} -gm_0\Gamma_e(\mathbf{y}) = \dfrac{\mu\sigma^4}{2}\Delta_{\mathbf{y}}\Gamma_e(\mathbf{y}) + \mathbf{v}\cdot\nabla_{\mathbf{y}}\Gamma_e(\mathbf{y}) + U_0(\mathbf{y})\Gamma_e(\mathbf{y}) + g\,m_e(\mathbf{y})\Gamma_e(\mathbf{y}) \\ -gm_0\Phi_e(\mathbf{y}) = \dfrac{\mu\sigma^4}{2}\Delta_{\mathbf{y}}\Phi_e(\mathbf{y}) - \mathbf{v}\cdot\nabla_{\mathbf{y}}\Phi_e(\mathbf{y}) + U_0(\mathbf{y})\Phi_e(\mathbf{y}) + g\,m_e(\mathbf{y})\Phi_e(\mathbf{y}) \end{cases}$$



Characteristic length and velocity scales

Intruder

- Radius: R
- Velocity: v

Characteristic length and velocity scales

Intruder

- Radius: R
- Velocity: v

Pedestrians

- Healing length: $\xi = \sqrt{|\mu\sigma^4/2gm_0|}$
- "Sound velocity": $c_s = \sqrt{|gm_0/2\mu|}$

Characteristic length and velocity scales

Intruder

- Radius: R
- Velocity: v

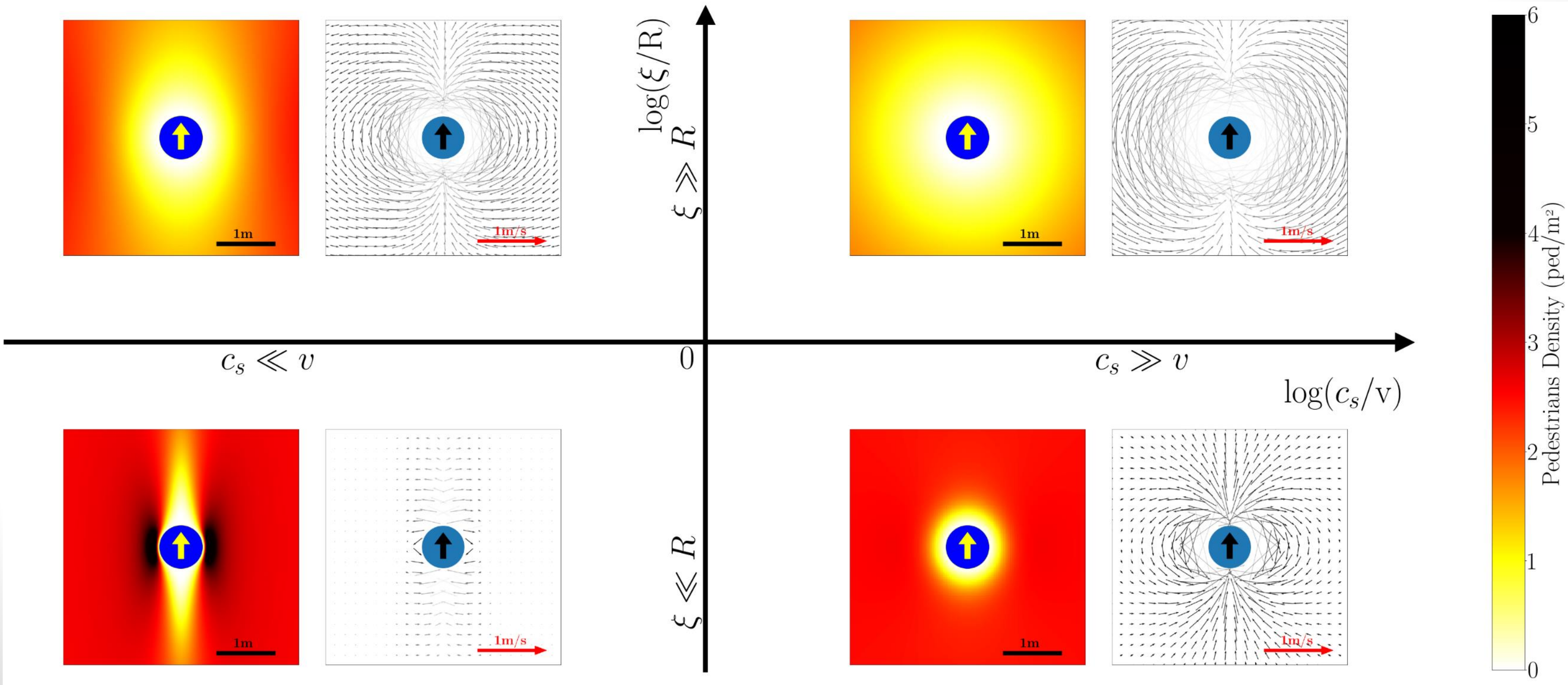
Pedestrians

- Healing length: $\xi = \sqrt{|\mu\sigma^4/2gm_0|}$
- "Sound velocity": $c_s = \sqrt{|gm_0/2\mu|}$

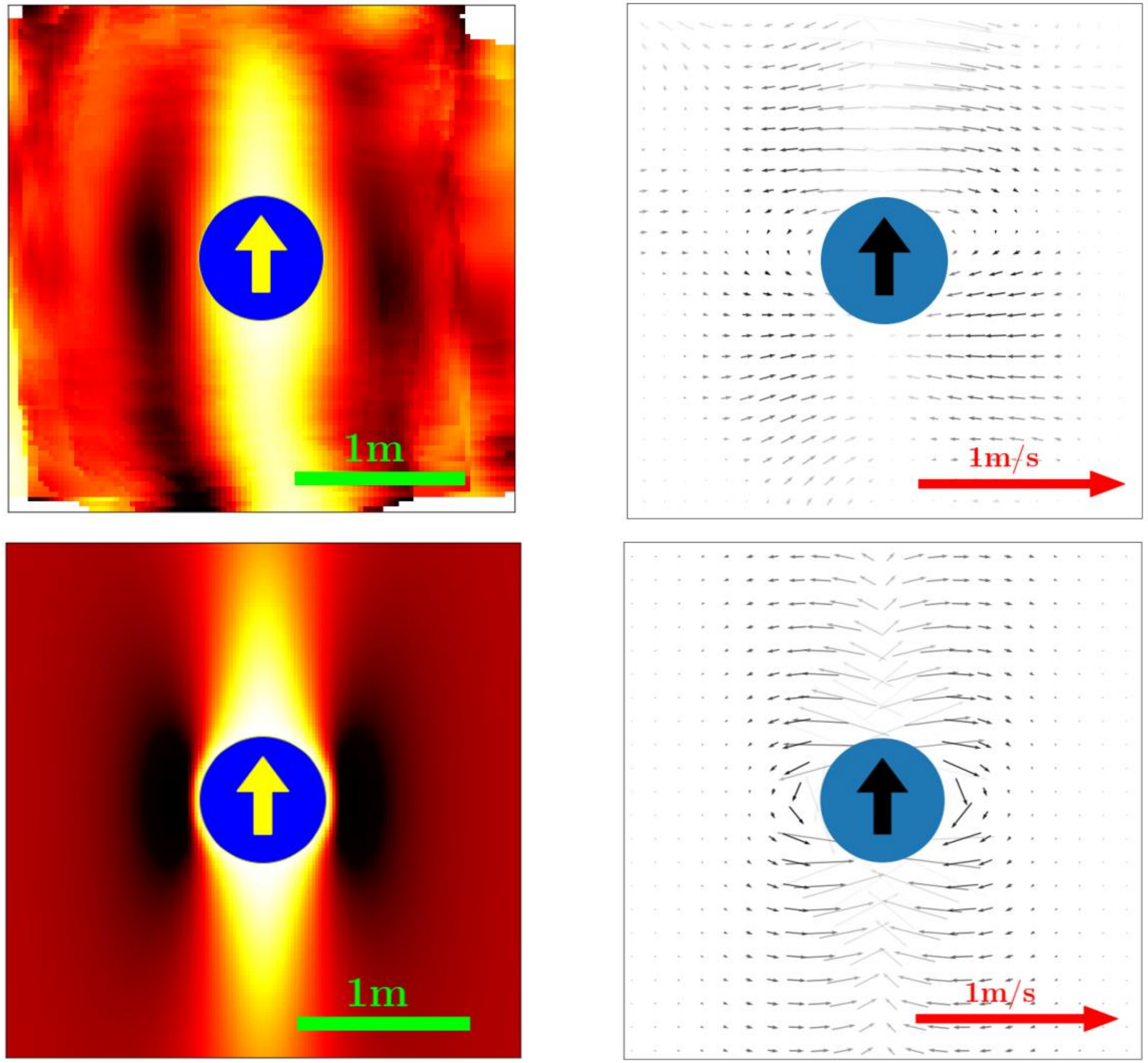
Up to a scaling factor, solutions of the (ergodic) MFG equations depend only on ξ/R and c_s/v.

Characteristic length and velocity scales

Up to a scaling factor, solutions of the (ergodic) MFG equations depend only on ξ/R and c_s/v.



Comparison to experiment



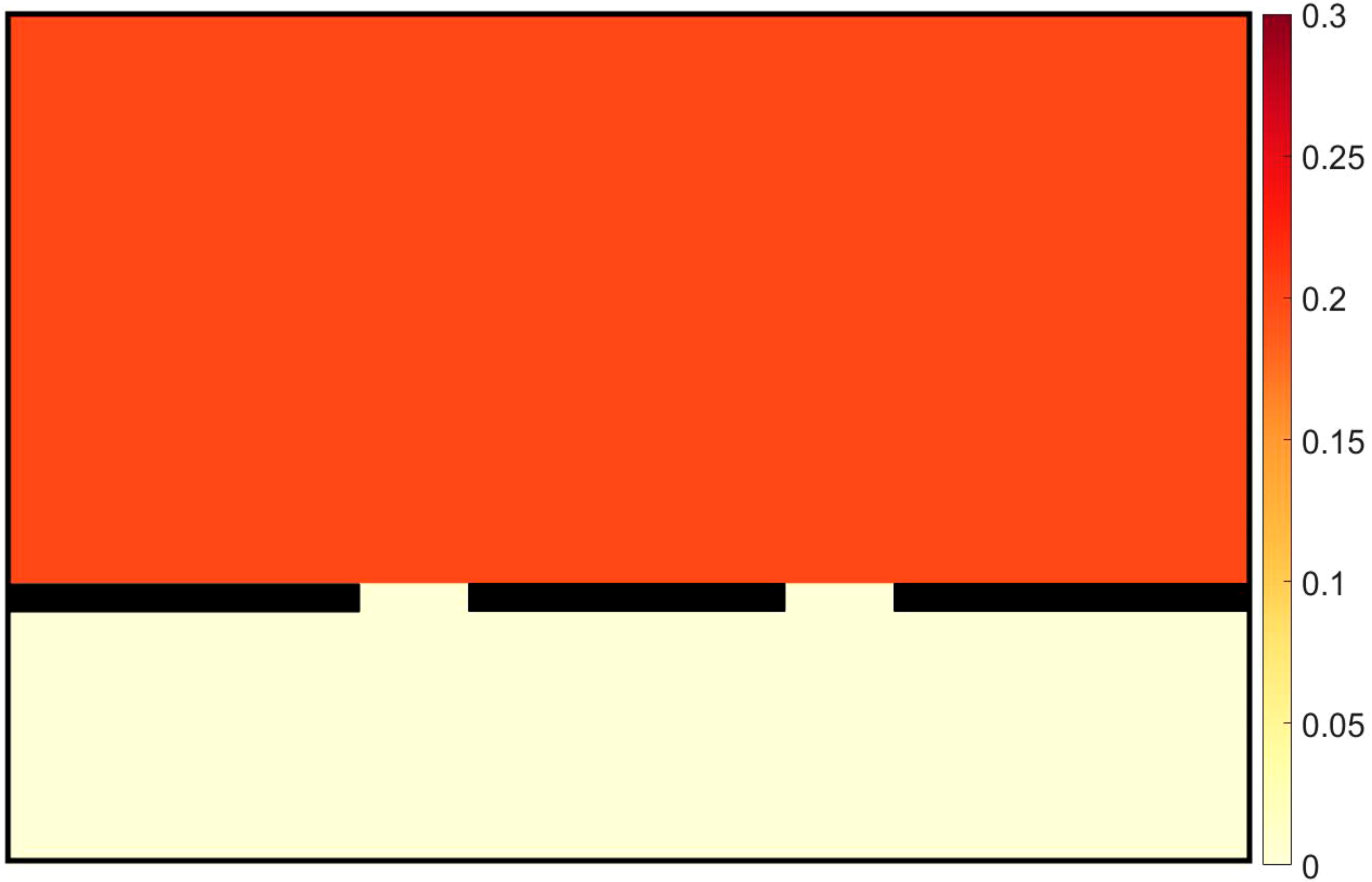
Good qualitative agreement.

Does better than the other models.

Cannot claim quantitative agreement.

Not limited to obstacle avoidance

[Bonnemain et al. PRE (2023)]



Discounted Mean-Field Games

[Butano, Appert-Rolland, Ullmo (2023)]

- Add a discount factor

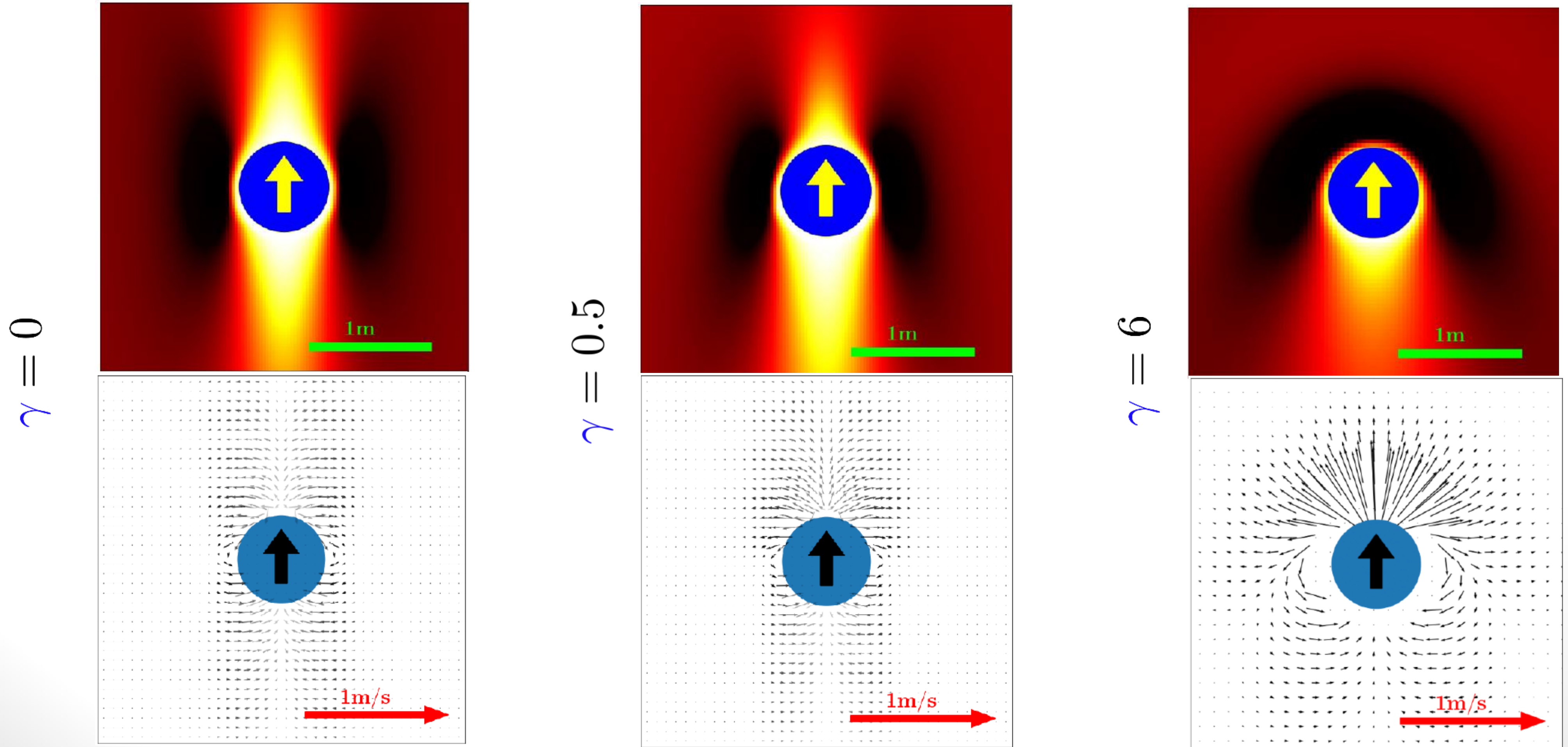
$$\mathbb{E}\left[\int_t^T \left(\frac{\mu}{2}(\mathbf{a}_s^i)^2 - g m(\mathbf{X}_s^i, s) - U_0(\mathbf{X}_s^i - \mathbf{v}t)\right) e^{\gamma(t-s)} \mathrm{d}s + c_T(\mathbf{X}_T^i) e^{\gamma(t-T)}\right]$$

Discounted Mean-Field Games

[Butano, Appert-Rolland, Ullmo (2023)]

- Add a discount factor

$$\mathbb{E}\left[\int_t^T \left(\frac{\mu}{2}(\mathbf{a}_s^i)^2 - g m(\mathbf{X}_s^i, s) - U_0(\mathbf{X}_s^i - \mathbf{v}t)\right) e^{\gamma(t-s)} \mathrm{d}s + c_T(\mathbf{X}_T^i) e^{\gamma(t-T)}\right]$$



Conclusions

- MFG reproduce qualitatively well the intruder experiment.
- Even in its simplest form does better than models found in commercial softwares (at least in this configuration).

Conclusions

- MFG reproduce qualitatively well the intruder experiment.
- Even in its simplest form does better than models found in commercial softwares (at least in this configuration).

But...

- We cannot yet pretend to quantitative accuracy.
- MFG will struggle to address features associated with the fact that pedestrians are discrete entities. Their rationality may also be an issue.

Conclusions

- MFG reproduce qualitatively well the intruder experiment.
- Even in its simplest form does better than models found in commercial softwares (at least in this configuration).

But...

- We cannot yet pretend to quantitative accuracy.
- MFG will struggle to address features associated with the fact that pedestrians are discrete entities. Their rationality may also be an issue.

And thankfully...

- MFG is versatile.
- Room for improvement (e.g. congestion [Lachapelle, Wolfram (2011)], discount factor, more accurate interaction potential, feedback on the intruder...).