M2 ICFP Internship : Theoretical Track

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Applications of perturbation methods to Mean Field Game

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1 Introduction

At the end of the last century, a new field of mathematics began to emerge : Game Theory.[23] This theory aims to describe strategic interactions between rational agents. It finds applications in various disciplines, including social science [1], finance [13], systems science, engineering [18], and economics. However, when dealing with a large number of agents whose behavior is subject to uncertainty described by a Langevin's equation, the complexity of the equations can become overwhelming.

Fortunately, during the same period, Statistical Physics discovered a powerful tool to simplify complex equations called the mean field approximation. By incorporating this approximation from Physics into the intricate equations of Game Theory, the field of Mean Field Games was born. Initially introduced by mathematicians Jean-Michel Lasry and Pierre-Louis Lions, this theory has recently generated significant excitement. In this report, we aim to introduce readers to this emerging field and explore the potential outcomes of applying perturbative methods to problems in mean field games.

2 State of Art

2.1 Prerequisites

2.1.1 Introduction to Game theory

Game Theory is a modern framework used to describe the process of strategic optimization. The term "strategic" indicates that each participant referred to as an agent or player, seeks to solve an optimization problem influenced by the actions or strategies of other participants. As the number of interactions increases, the complexity escalates, necessitating a clear and precise definition of what constitutes a solution to the optimization problem.[22]

Different types of games exist and the game that we are going to play is :

- non-cooperative : agents focus on individual gains rather than look for a collective greater pay-off.
- sequential : strategy of each player can change over time.
- rational : agents are able to solve optimization problems and follow their strategy without psychological bias.

In order to better visualize what these rules could mean, let's see the basic example of a game with two agents. The most famous one is the prisoner's dilemma described by William Poundstone in his book[24] :

Two members of a criminal gang, A and B, have been apprehended and are now imprisoned separately. Each prisoner is in solitary confinement without any means of communication

with their partner. The primary accusation carries a ten-year prison sentence, but the police lack sufficient evidence for a conviction. Instead, they plan to sentence both prisoners to two years in prison on a lesser charge. However, they present each prisoner with a moral dilemma : If one of them confesses to the principal charge, thus betraying the other, they will receive a pardon and be released, while the other will bear the full sentence of ten years instead of just two years for the lesser charge.

This leads to four different possible outcomes :

- A : If A and B both remain silent, they will each serve the lesser charge of 2 years in prison.
- B : If A betrays B but B remains silent, A will be set free while B serves 10 years in prison.
- C : If A remains silent but B betrays A, A will serve 10 years in prison and B will be set free.
- D : If A and B both betray the other, they will share the sentence and serve 5 years.

It is assumed that both players lack loyalty towards each other and possess a clear understanding of the dynamics of this game. The dilemma arises from the fact that mutual betrayal leads to a worse outcome compared to mutual cooperation (resulting in a combined sentence of 10 years instead of 4). However, from a self-interested perspective, cooperating appears irrational. Presented in this manner, betraying always results in a better payoff than remaining silent, regardless of the other player's decision. If B betrays A, it is logical for A to retaliate and betray B, as facing a conviction for 5 years is preferable to 10 years.
And if B stays silent, A should still betray B as going free is better than serving 2 years. Because of this, betraying is called a dominant strategy and the situation where both prisoners betray the other is referred to as a Nash Equilibrium.

This canonical example of a game is a perfect illustration of a non-cooperative, non-sequential, and rational game.

In the realm of Game Theory, solutions refer to predictions regarding the strategies that players will adopt and the resulting outcome of the game. Naturally, the outcome of the game can vary depending on the level of rationality exhibited by the players, as defined within the framework of game theory. This variability has led to the development of various solution concepts, with the Nash Equilibrium concept being the most commonly employed.

The Nash Equilibrium concept assumes that players will select their strategies in a manner that leads to a state known as Nash Equilibrium.[28]

Consider a non-cooperative game with N rational players. Let s_i be a strategy adopted by player i from a compact metric space S_i in order to optimize a cost function c_i, continuous

and real-valued on $\prod_{j=1}^{N} s_j$. A Nash Equilibrium is then a N-tuple $(\bar{s}_1, \ldots, \bar{s}_N) \in \prod_{j=1}^{N} S_j$ such that, for any $i = 1, \ldots, N$:

$$c_i\left(\bar{s}_1, \ldots, \bar{s}_N\right) \leq c_i\left(s_i, \left(\bar{s}_j\right)_{j \neq i}\right), \quad \forall s_i \in S_i$$

Indeed, in a Nash Equilibrium, no player has a motivation to unilaterally deviate from their chosen strategy since such a change would lead to a higher cost or a suboptimal outcome for that player. It represents a stable state where each player's strategy is the best response to the strategies chosen by the other players, given their rationality and knowledge of the game.

However, it is indeed important to acknowledge that Nash Equilibrium is not always present in every game, and there can be situations where multiple Nash Equilibria exist. Despite these limitations, Nash Equilibrium remains a widely employed and reasonable solution concept for analyzing a broad range of games. In our discussion, we will assume that the games we consider have at least one Nash Equilibrium to facilitate our analysis.

2.1.2 Differential games

In N-players differential game, each player (assumed to be rational) is characterized by a state vector $\overrightarrow{X}_t^i \in \mathbb{R}^n$ representing their current state, and their strategy a_t^i, which is dynamically adjusted as the game unfolds. To capture the inherent uncertainty in a player's decision-making process, one can model their behavior using Langevin dynamics.

$$d\overrightarrow{\mathrm{X}}_t^i = \overrightarrow{a}_t^i dt + \sigma_i d\overrightarrow{W}_t^i \tag{1}$$

In this context, $\overrightarrow{W}_t^i$ represents a Gaussian white noise process that is independent of $\overrightarrow{W}^{j \neq i}$ and has an amplitude of 1. Each player's objective is to optimize a cost functional c_t^i that takes into account the behavior of every other player in the game. The strategies chosen by the players influence the dynamics of the game and impact the overall cost incurred by each player.

$$c_t^i\left[\overrightarrow{a}^1, \ldots, \overrightarrow{a}^N\right]\left(\overrightarrow{X}_t^1, \ldots, \overrightarrow{X}_t^N\right) = \mathbb{E}\left[\int_t^T \left(c^i(\overrightarrow{X}_\tau^i, \overrightarrow{a_\tau^i}) - V^i\left(\overrightarrow{X}_\tau^1, \ldots, \overrightarrow{X}_\tau^N\right)\right) d\tau + c_T^i\left(\overrightarrow{X}_T^1, \ldots, \overrightarrow{X}_T^N\right)\right]$$

— with $\mathbb{E}$ the average on the noise.

— T is the time at which the game ends.

— the mobile cost c^i strictly convex on a^i (the greater the drift, the more expensive it is to increase it). The mobile cost represents how much a strategy is expensive to realize.

— V represents the environmental gain, which can be understood as a measure of the overall benefit obtained from the controlled dynamical system.
e.g. : a well potential that favors players located in a specific zone of the space. In a realistic situation, people at a concert want to be in the front of the scene.

— c_T represents the terminal cost, which represents the cost associated with concluding the control process in a particular configuration or state. This terminal cost is incurred at the final time T and is taken into account when evaluating the overall performance or optimization of the system.
e.g. : In the case of the well potential explained above, if the terminal cost associated with finishing in this zone is too high, you will observe that close to the end of the game, all players will leave the zone because of this cost.

In my research, I have specifically considered cost functions that are based on the square of the control parameter.

If we postulate all players have quadratic running cost the cost functional become :

$$c_t^i\left[\overrightarrow{a}^1,\ldots,\overrightarrow{a}^N\right]\left(\overrightarrow{X}_t^1,\ldots,\overrightarrow{X}_t^N\right)=\mathbb{E}\left[\int_t^T\left(\frac{\mu^i}{2}\left(\overrightarrow{a}_\tau^i\right)^2-V^i\left(\overrightarrow{X}_\tau^1,\ldots,\overrightarrow{X}_\tau^N\right)\right)d\tau+c_T^i\left(\overrightarrow{X}_T^1,\ldots,\overrightarrow{X}_T^N\right)\right]$$

— where μ^i is a constant.

It is worth noting that a natural correspondence can be observed between the quadratic term and the kinetic energy in a physical system. In this analogy, the faster you run, the more energy it requires to increase your speed. Similarly, when deviating from the optimal strategy in a game, the cost of moving further away from that strategy increases.

2.1.3 Interlude on optimal control

Optimal Control is a discipline within applied mathematics that deals with the analysis of dynamic systems capable of being influenced and controlled to optimize a particular objective or goal.[8] It can be seen as a contemporary extension of the well-established field of Calculus of Variations, which finds extensive applications in physics. Given the inherent link between optimization and Game Theory, Optimal Control forms a strong basis for the development of Mean Field Games (MFG) theory. The principles and techniques of Optimal Control play a vital role in analyzing and comprehending the dynamics and strategies inherent in MFGs.

Let us consider a system specified by its state variable $\overrightarrow{\mathrm{X}} \in \mathbb{R}^d$ which evolves according to Langevin dynamics :

$$d\overrightarrow{X}_t=\overrightarrow{a}_t dt+\sigma d\overrightarrow{W}_t \tag{2}$$

where $\overrightarrow{a}$ is the control parameter, σ a constant and $\overrightarrow{W}$ a Gaussian white noise of variance one.

We want to control (i.e. find the proper dynamics for $\overrightarrow{a}$) so that the following cost function is minimized :

$$c[\overrightarrow{a}](\overrightarrow{X},t) = \mathbb{E}\left[\int_t^T \left(\frac{\mu}{2}\left(\overrightarrow{a}_\tau\right)^2 - V\left(\overrightarrow{X}_\tau\right)\right)d\tau + c_T\left(\overrightarrow{\mathrm{X}}_T\right)\right] \tag{3}$$

— where μ is a constant.

In order to find the Optimal Control parameter $\overrightarrow{a}*$, let us first introduce the value function u as the optimal cost function

$$u(\overrightarrow{X},t) \equiv \inf_{\overrightarrow{a}} c[\overrightarrow{a}](\overrightarrow{X},t) \tag{4}$$

To determine the value of u and establish its connection to $\overrightarrow{a}^*$, we employ the concept of dynamic programming, specifically Bellman's optimality principle [3]. This principle is a fundamental concept in dynamic programming that allows us to solve optimization problems by breaking them down into smaller subproblems.

"An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision."

This means the computation of u can be simplified by discretizing the problem and optimizing on infinitesimally small time frames.

On the interval $[t, t+dt]$ we then have the relation :

$$u(\overrightarrow{\mathrm{X}},t) = \inf_{\overrightarrow{a}} \mathbb{E}\left[\left[\int_t^{t+dt}\left(\frac{\mu}{2}\left(\overrightarrow{a}_\tau\right)^2 - V\left(\overrightarrow{\mathrm{X}}_\tau\right)\right)d\tau\right] + u(\overrightarrow{\mathrm{X}} + d\overrightarrow{\mathrm{X}}, t+dt)\right] \tag{5}$$

which is called the Bellman's equation. The dynamics of $\overrightarrow{X}$ being defined through Langevin equation, $u(\overrightarrow{\mathrm{X}} + d\overrightarrow{\mathrm{X}}, t+dt)$ can be expanded to first order in dt using Itô's Lemma [15] :

$$\mathbb{E}[u(\overrightarrow{\mathrm{X}} + d\overrightarrow{\mathrm{X}}, t+dt)] = u(\overrightarrow{\mathrm{X}},t) + \left[\partial_t u(\overrightarrow{\mathrm{X}},t) + \overrightarrow{a}\cdot\overrightarrow{\nabla}u(\overrightarrow{\mathrm{X}},t) + \frac{\sigma^2}{2}\Delta u(\overrightarrow{\mathrm{X}},t)\right]dt \tag{6}$$

which, combined with Bellman's equation (5) yields Hamilton-Jacobi-Bellman's (HJB) equation :

$$\partial_t u + \frac{\sigma^2}{2}\Delta u + \inf_{\overrightarrow{a}}\left[\frac{\mu}{2}(\overrightarrow{a})^2 + \overrightarrow{a}\cdot\overrightarrow{\nabla}u\right] = V \tag{7}$$

The optimization of the third term in the Hamilton-Jacobi-Bellman's (HJB) equation (7) leads to the relation $\overrightarrow{a}^* = -\frac{\overrightarrow{\nabla}u}{\mu}$.The negative sign in the relation $\overrightarrow{a}^* = -\frac{\overrightarrow{\nabla}u}{\mu}$ indicates that

the optimal control action is in the opposite direction of the gradient of the value function.

This means that the control action should be chosen to move in the direction that decreases the value function the fastest. Then HJB's equation becomes :

$$\partial_t u + \frac{\sigma^2}{2}\Delta u - \frac{1}{2\mu}\|\overrightarrow{\nabla} u\|^2 = V \tag{8}$$

It is important to note that the HJB's equation is formulated in a backward-in-time manner. This is indicated by the sign in front of the diffusive term. In this formulation, the boundary condition for the value function is specified at the final time T, which marks the end of the optimization process.

$$u(\overrightarrow{\mathrm{X}}, T) = c_T(\overrightarrow{\mathrm{X}}) \tag{9}$$

and the solution to HJB's equation is then constructed step by step from there.

This concludes this introduction to Optimal Control, we now have the background to pursue our journey.

2.1.4 Return on differential games

If we postulate all players have quadratic running costs as mentioned at the end of Section 2.1.2 and because we consider rational players, we look for a Nash Equilibrium $\left(\overrightarrow{a}^1*, \ldots, \overrightarrow{a}^N*\right)$ as the solution of this optimization problem.

$$c^i\left[\overrightarrow{a}^1*, \ldots, \overrightarrow{a}^N*\right] \leq c^i\left[\overrightarrow{a}^1*, \ldots, \overrightarrow{a}^i, \ldots, \overrightarrow{a}^N*\right] \quad \forall i = 1, \ldots, N \text{ and } \forall \overrightarrow{a}^i \tag{10}$$

In this context, the value function u_t^i may be defined as the optimal cost of player i if every other player follows their optimal strategy :

$$u_t^i\left(\overrightarrow{X}_t^1, \ldots, \overrightarrow{X}_t^N\right) = \inf_{\overrightarrow{a}^i} \mathbb{E}\left[\int_t^T \left(\frac{\mu^i}{2}\left(\overrightarrow{a}_\tau^i\right)^2 - V^i\left(\overrightarrow{X}_\tau^1, \ldots, \overrightarrow{\mathrm{X}}_\tau^i, \ldots \overrightarrow{\mathrm{X}}_\tau^N\right)\right) d\tau + c_T^i\left(\overrightarrow{\mathrm{X}}_T^1, \ldots, \overrightarrow{\mathrm{X}}_T^i, \ldots, \overrightarrow{X}_T^N\right)\right]$$

and evolves according to the following HJB's equation [19] :

$$\partial_t u^i + \frac{\sigma_i^2}{2}\sum_{j=1}^N \Delta_{x^j} u^i - \sum_{j\neq i}\frac{1}{\mu^j}\left(\overrightarrow{\nabla}_{x^j} u^j\right)\cdot\left(\overrightarrow{\nabla}_{x^j} u^i\right) - \frac{1}{2\mu^i}\left\|\overrightarrow{\nabla}_{x^i} u^i\right\|^2 = V^i \tag{11}$$

with the terminal condition :

$$u_T^i\left(\overrightarrow{X}_T^1, \ldots, \overrightarrow{\mathrm{X}}_T^N\right) = c_T^i\left(\overrightarrow{X}_T^1, \ldots, \overrightarrow{\mathrm{X}}_T^N\right) \tag{12}$$

As the number of players in a differential game grows, the complexity of the problem escalates significantly due to the increasing number of interrelated stochastic equations. This growth presents challenges in terms of analyzing and finding solutions for the game, making

it computationally intractable. Consequently, there is a demand for a novel framework that can effectively handle a large number of players in a more tractable manner.

To tackle this challenge, the field of Mean Field Games (MFG) has emerged as a promising framework. MFG offers a tractable approximation of the behavior exhibited by a large population of interacting players. It achieves this by introducing simplifying assumptions that enable the analysis of game dynamics at a mean field level. In this approach, the individual behaviors of players are replaced by an aggregate behavior, allowing for more manageable analysis and solution methods.

2.2 Mean Field approach

2.2.1 Introduction to MFG

Mean Field Games were first introduced by P.-L. Lions and J.-M. Lasry [20] [21] and further developed by M. Huang, R. P. Malhamé, and P. E. Caines [17]. They were designed as a framework to tackle non-atomic differential games involving a significant number of identical players. The concept of mean field, drawing inspiration from its counterpart in physics [14], plays a central role in MFG by simplifying the analysis. Within this framework, agents are assumed to be insensitive to the individual choices of others but are influenced by an aggregated quantity that represents the decisions of all players.

Various approaches to Mean Field Games have been extensively investigated in recent years. In this report, my primary focus will be on the deterministic Partial Differential Equation (PDE) formulation initially proposed by P.-L. Lions and J.-M. Lasry.

To formulate the Mean Field Games equations, we begin by incorporating the aforementioned assumptions into the framework of differential games described in section 2.1.4. We consider a group of identical players who share common characteristics such as $\sigma_i = \sigma$, $\mu^i = \mu$, $V^i = V$, and $c_T^i = c_T$. The only differences among the players are their initial states X_0^i and choice of strategy a_t^i. Furthermore, we assume that the potential V and the terminal cost c_T experienced by an individual player i depend only on the collective behavior of all players, which is represented by the empirical density :

$$\tilde{m}(x,t) = \frac{1}{N}\sum_{j=1}^{N} \delta\left(x - X_t^j\right) \tag{13}$$

Hence $V\left(\overrightarrow{X}_t^1, \ldots, \overrightarrow{X}_t^N\right) \approx V[\tilde{m}]\left(X_t^i\right)$ and $c_T\left(\overrightarrow{X}_T^1, \ldots, \overrightarrow{X}_T^N\right) \approx c_T[\tilde{m}]\left(X_T^i\right)$.
As we consider a large number of players and assume that the fluctuations of the empirical density $\tilde{m}$ can be neglected, we can make use of the mean field approximation. In this

approximation, we introduce a mean field $m(x,t)$, which represents the average of the empirical density over all realizations of the noise. Based on this mean field, we define the value function for a particular player as in [10] :

$$u(x,t) = \inf_{\overrightarrow{a}} \mathbb{E}\left[\int_t^T \left(\frac{\mu}{2}\left(\overrightarrow{a}^i_\tau\right)^2 - V^i[m]\left(\overrightarrow{X}^i_\tau\right)\right) d\tau + c^i_T[m]\left(\overrightarrow{X}^i_T\right)\right] \tag{14}$$

This allows us to recast the differential game presented in section 2.1.4 into a one-body optimization problem as described in section 2.1.2 while the agents essentially decouple. The value function then verifies HJB's equation (7) :

$$\begin{cases} \partial_t u + \frac{\sigma^2}{2}\Delta u - \frac{1}{2\mu}\|\overrightarrow{\nabla} u\|^2 = V[m] \\ u(x,T) = c_T[m](x) \end{cases} \tag{15}$$

We retrieve the same equation (8) of section 2.1.4 but now V depends on the mean field m.

Similar to the field of Physics, the introduction of a mean field allows for the decoupling of players' dynamics in Mean Field Games (MFG). However, it is crucial to ensure self-consistency within the system. This can be achieved by acknowledging that when a sufficiently large number of players adhere to Langevin dynamics (1), the density of players m (which also serves as the mean field) can be effectively described using the Fokker-Planck (FP) equation [25]. The FP's equation provides a mathematical framework to capture the evolution of the density of players as they interact and make decisions within the game.

$$\partial_t m + \overrightarrow{\nabla} \cdot [m\,\overrightarrow{a}_*] - \frac{\sigma^2}{2}\Delta m = 0 \tag{16}$$

where $\overrightarrow{a}_* = -\frac{1}{\mu}\overrightarrow{\nabla} u$ is the optimal control parameter according to HJB's equation.
Considering an initial distribution of players $m_0(x)$ the MFG problem reduces to a system of coupled (deterministic) PDE's :

$$\begin{cases} \partial_t u + \frac{\sigma^2}{2}\Delta u - \frac{1}{2\mu}\|\overrightarrow{\nabla} u\|^2 = V[m] \\ u(\overrightarrow{x}, t=T) = c_T[m](\overrightarrow{x}) \\ \partial_t m - \frac{1}{\mu}\overrightarrow{\nabla} \cdot [m\overrightarrow{\nabla} u] - \frac{\sigma^2}{2}\Delta m = 0 \\ m(\overrightarrow{x}, t=0) = m_0(\overrightarrow{x}) \end{cases} \tag{17}$$

In this formulation, the emphasis is placed on the mean field $m(x,t)$ and the value function $u(x,t)$ as the primary unknowns, shifting away from individual strategies. It is noteworthy that the system exhibits an unconventional forward-backward structure. The Hamilton-Jacobi-Bellman's equation is derived from the terminal cost at the end of the game, while the Fokker-Planck's equation describes the evolution of the player distribution from its initial

configuration.

The presence of mixed initial-final boundary conditions introduces new challenges in characterizing solutions to MFG's equations, both analytically and numerically. This aspect can lead to dynamics that may deviate from typical physical scenarios. As a result, the analysis and numerical treatment of MFGs require innovative approaches to overcome these challenges and accurately capture the complex dynamics of the system.

2.2.2 Existence of an ergodic state

In sufficiently long games, it is conceivable that there exists a time, which is distant from both the beginning and the end of the game, where the solutions to the MFG's equations (17) become completely decoupled from the initial and final conditions. At this point, players no longer need to remember their initial state or be concerned about the endgame, but can still devise an optimal, potentially stationary, strategy.

In line with this notion, P. Cardaliaguet et al. [11] demonstrated that in the limit as T approaches infinity, under the condition that the potential $V[m](x)$ does not explicitly depend on time (which we consider throughout this manuscript), and if the system is confined in some manner (either through V or due to fixed spatial boundaries), there exists an ergodic state such that :

$$\begin{cases} m(\overrightarrow{x}, t) \simeq \bar{m}(\overrightarrow{x}) \\ u(\overrightarrow{x}, t) \simeq \bar{u}(\overrightarrow{x}) + \lambda t \quad (\text{for } 0 \ll t \ll T) \end{cases} \tag{18}$$

The constant λ can be determined through the normalization of m. This finding holds significant importance as it provides a broad understanding of how a game will unfold, even if the system of forward-backward equations (17) cannot be fully solved. It simplifies the problem by eliminating the time dependence, allowing for a clearer analysis of the game dynamics. Leading to the following system :

$$\begin{cases} \lambda + \frac{\sigma^2}{2}\Delta\bar{u} - \frac{1}{2\mu}\|\overrightarrow{\nabla}\bar{u}\|^2 = V[m] \\ \frac{1}{\mu}\overrightarrow{\nabla} \cdot [\bar{m}\overrightarrow{\nabla}\bar{u}] + \frac{\sigma^2}{2}\Delta\bar{m} = 0 \end{cases} \tag{19}$$

3 Link with known physics

While the system of MFG's equations (17) offers a notable simplification compared to the differential games discussed in section 2.1.4, solving them still presents challenges. The primary difficulty arises from the forward-backward nature of these equations, which is not commonly encountered in physics and introduces its own complexities.

However, in the specific case of quadratic MFG, there are approaches available to reformulate the problem into a more familiar framework for physicists. These alternative forms

of the MFG's equations (17) provide a different perspective and can potentially facilitate the analysis and solution of the problem. In the ensuing discussion, I will explore these alternative formulations of the MFG's equations (17).

3.1 Nonlinear Schrodinger's equation

The change of variable mentioned, which was initially introduced by O. Guéant as a means to construct a monotonous discretization scheme [16], has been extensively discussed by D. Ullmo, I. Swiecicki, and T. Gobron [30]. It serves as a gateway for physicists to delve into the realm of Mean Field Games (MFG) and provides a framework that aligns with their familiar approaches and techniques. The utilization of this change of variable aids in addressing the challenges posed by the forward-backward structure of MFG equations and allows for the application of methods more commonly employed by physicists. [29] [6]

Proceeding as in [30], one can make use of the classic Cole-Hopf transform on HJB equation to obtain a standard heat equation :

$$\begin{cases} \mu\sigma^2 \partial_t \Phi = -\frac{\mu\sigma^4}{2} \Delta\Phi - V[m]\Phi \\ u = -\mu\sigma^2 \log \Phi \end{cases} \tag{20}$$

where this equation is constructed backward in time, similarly to the HJB's equation, with the terminal condition : $\Phi(x, t = T) = \exp\left[-c_T(x)/\mu\sigma^2\right]$.

One can then perform a "hermitization" of equations (17) :

$$\begin{cases} \mu\sigma^2 \partial_t \Gamma = \frac{\mu\sigma^4}{2} \Delta\Gamma + V[m]\Gamma \\ m = \Phi\Gamma \end{cases} \tag{21}$$

this one being forward in time with initial condition $\Gamma(x, t = 0) = m_0(x)/\Phi(x, 0)$.

Through these transformations the system of MFG's equations (17) exhibits a mapping onto the Non-linear Schrödinger's (NLS) equation :

$$\begin{cases} i\hbar \partial_t \Psi = -\frac{\hbar^2}{2\mu} \Delta\Psi - V[\rho]\Psi \\ i\hbar \partial_t \Psi^* = \frac{\hbar^2}{2\mu} \Delta\Psi^* + V[\rho]\Psi^* \end{cases} \tag{22}$$

under the formal correspondence :

— $\mu\sigma^2 \to \hbar$.

— $\Phi(x, t) \to \Psi(x, it)$ and $\Gamma(x, t) \to [\Psi(x, it)]^*$.

— $\rho \equiv \|\Psi\|^2 \to m \equiv \Phi\Gamma$.

The system of equations represented by (20) and (21) indeed exhibits differences when compared to the Nonlinear Schrödinger's (NLS) equation. Firstly, it retains the characteristic forward-backward structure that is inherent in Mean Field Games (MFGs). This structure distinguishes MFG's from other equations encountered in physics.

Secondly, the functional space in which the solutions Φ and Γ reside differs from the typical functional spaces encountered in physics. As physicists, we are familiar with working in various types of functional spaces, each with its own properties and challenges. In the context of MFG equations, the choice of functional space introduces unique characteristics and complexities that need to be carefully considered in the analysis and solution of the equations.

The construction of Φ and Γ in the Mean Field Games's equations specifies them as non-periodic, positive functions, while Ψ is allowed to be complex-valued. These distinctions from the Nonlinear Schrödinger's equation are indeed noteworthy. However, it is important to recognize that these differences do not diminish the significance of the mapping between the two equations.

The NLS's equation has been extensively studied in various fields, including non-linear optics, Bose-Einstein condensation, and fluid dynamics. Over the years, numerous methods have been developed to tackle the NLS's equation, and many of these methods can be adapted and applied to MFG models. This highlights the potential for leveraging the insights and techniques from the study of NLS's equations to enhance our understanding and analysis of MFG models, as emphasized in [30].

Now our system looks like this :

$$\begin{cases} -\mu\sigma^2\partial_t\Phi = \frac{\mu\sigma^4}{2}\Delta\Phi + V[m]\Phi \\ \mu\sigma^2\partial_t\Gamma = \frac{\mu\sigma^4}{2}\Delta\Gamma + V[m]\Gamma \\ \Phi(x, t = T) = \exp\left[-c_T(x)/\mu\sigma^2\right] \\ m(\overrightarrow{x}, t = 0) = m_0(\overrightarrow{x}) \end{cases} \tag{23}$$

3.2 Action and conserved quantities

The alternative representations of Mean Field Games's equations, as discussed earlier, offer the advantage of enabling the application of various methods and concepts originally developed in physics to study and characterize MFG problems. One notable aspect is the introduction of the concepts of action and energy into the context of MFG. These notions, widely utilized in physics, bring valuable insights and tools for analyzing and understanding MFG systems. [4]

The concepts of action and energy are fundamental in physics and have been extensively studied and applied in various physical systems. By incorporating these concepts into the

study of MFG, we gain a deeper understanding of the dynamics and behavior of the system. This cross-pollination of ideas between physics and MFG enriches the theoretical framework and opens up new avenues for analysis and interpretation. [7]

The system of equations (23) can be seen as deriving from an action S defined as :

$$S[\Gamma, \Phi] \equiv \int_0^T dt \int_{\mathbf{R}} dx \left[\frac{\mu\sigma^2}{2} \left(\Gamma \partial_t \Phi - \Phi \partial_t \Gamma \right) - \frac{\mu\sigma^4}{2} \nabla\Gamma \cdot \nabla\Phi + U[m] \right] \tag{24}$$

where $U[m]$ represents the functional anti-derivative of $V[m]$, so its minimization yields Schrodinger's representation of MFG's equations.

$$\begin{cases} \frac{\delta S}{\delta \Gamma} = 0 \\ \frac{\delta S}{\delta \Phi} = 0 \end{cases} \Leftrightarrow \begin{cases} \mu\sigma^2 \partial_t \Phi = -\frac{\mu\sigma^4}{2} \Delta\Phi - V[m]\Phi \\ \mu\sigma^2 \partial_t \Gamma = \frac{\mu\sigma^4}{2} \Delta\Gamma + V[m]\Gamma \end{cases}$$

The existence of an underlying action in MFG dynamics has a significant implication. Due to the time translation invariance of the integrand in (24), the Noether theorem guarantees the existence of a conserved quantity, analogous to energy in physical systems.

$$E = \int_{\mathbf{R}} dx \left[-\frac{\mu\sigma^4}{2} \nabla\Gamma \cdot \nabla\Phi + U[m] \right] \tag{25}$$

Continuing with the analogy to physical systems, it is indeed possible to interpret the first term in each integrand, which depends on σ, as kinetic energy, while the term involving U corresponds to potential energy. However, it is crucial to recognize that the interpretation of these quantities differs in the context of social or engineering sciences compared to their traditional physical interpretation. In the context of MFG and related disciplines, these quantities should primarily be understood as abstract quantities that serve to formally characterize a problem.

4 Schelling Model

4.1 Historical context

The Schelling model is a social science model that aims to explain the phenomenon of geographical segregation [27]. It shares some similarities with well-known physical models. The model works as follows :

— We start with a square-based network where two different populations are randomly distributed. There are also free spaces on the network.

— Each agent has a cost function that captures their preference for being surrounded by similar individuals. The cost function increases when the proportion of neighbors belonging to the same population falls below a certain threshold $p \in [0, 1]$.

— Let $n^a_{i,j}$ denote the number of neighbors of site (i,j) belonging to population $a \in \{-1,1\}$. The cost function for an agent of population a at site (i,j) is given by :

$$c^a_{i,j} = \left(p - \frac{n^a_{i,j}}{n^a_{i,j} + n^{-a}_{i,j}}\right) \cdot \theta\left(p - \frac{n^a_{i,j}}{n^a_{i,j} + n^{-a}_{i,j}}\right) \tag{26}$$

where $\theta(x)$ is the Heaviside step function.

— At each iteration, unsatisfied agents, whose cost exceeds a certain threshold, are allowed to move to a free site in the network. The movement aims to improve their satisfaction by finding a neighborhood that better matches their preferences.

The Schelling model's conclusion, supported by simulations and analysis, is that even with relatively low preference thresholds (p), the system exhibits rapid convergence towards a state of segregation. This means that populations tend to cluster in separate blocks rather than being evenly distributed.

This finding has had a significant impact on contemporary sociological thinking. Prior to Schelling's work, the observation of geographical separation between communities often led to the assumption that populations were inherently racist or had a strong preference for homogeneity, actively seeking to avoid living near individuals from other populations. This perspective often led to policies and interventions aimed at reducing social mixing and promoting integration.

More generally, his work showed an inherent property of complex systems in a physical sense. In his original article [27], Schelling wrote about such social experiments that "lead to aggregate results that the individual neither intends nor needs to be aware of, results that sometimes have no recognizable counterpart at the level of the individual". This is not completely unknown for statistical physicists, it reminds of the famous formula from P.W. Anderson "More is different".[2]

However and as Shelling's stated in his paper, his explanation of segregation may be at the third place for explaining why there is segregation, and many other factors like economic segregation are dominant.

Schelling's insights have prompted a rethinking of the causes and dynamics of segregation, emphasizing the importance of systemic factors and the unintended consequences of individual behavior. This has influenced discussions and approaches to urban planning, social policy, and understanding of social dynamics in diverse communities.

In the next section, we will try to cast this model in a MFG formulation.

4.2 Casting into an MFG model

With what we learned from the previous section we can generalize our system (17) in the case of 2 populations interacting ($i \in \{-1, 1\}$) :

$$\begin{cases} -\partial_t u_i - \dfrac{\sigma_i^2}{2}\Delta u_i + \dfrac{1}{\mu_i}\nabla u_i = V_i\left[m_1, m_2; x\right] \\ \partial_t m_i - \dfrac{1}{\mu_i}\nabla\left((\nabla u) m_i\right) = \dfrac{\sigma_i^2}{2}\Delta m_i \end{cases} \tag{27}$$

and

$$\begin{cases} -\hbar_i \partial_t \Phi_i = \dfrac{\hbar_i^2}{2\mu_i}\Delta\Phi_i + V_i\left[m_1, m_2; x\right]\Phi_i \\ \hbar_i \partial_t \Gamma_i = \dfrac{\hbar_i^2}{2\mu_i}\Delta\Gamma_i + V_i\left[m_1, m_2; x\right]\Gamma_i \end{cases} \tag{28}$$

The first problem we face is that of choosing the cost function. We could simply adapt the function given in (26), but a potential of the form :

$$V_i\left[m_1, m_2, ; x\right] = \left(p - \frac{m_i(x)}{m_i(x) + m_{3-i}(x)}\right) \cdot \theta\left(p - \frac{m_i(x)}{m_i(x) + m_{3-i}(x)}\right) \tag{29}$$

does not allow us to simply write an action from which our system is derived. In this case, a numerical approach is preferred, however, we chose to work in an analytical framework. Hence we will adapt this function to a simpler one.

4.3 Limitation of the model

To simplify calculations and to get closer to a physical system, we will simplify the model : we will consider that agents aim to minimize the following potential :

$$V_i\left[m_1, m_2, ; x\right] = -g m_{3-i}(x) \tag{30}$$

/ with g a coupling constant.

In a sociological sense, this model is less interesting : the agents openly avoid strangers, and there's nothing revolutionary to be expected from large-scale segregation in such a system. However, this problem provides us with an example of what perturbative treatment can shed light on a Mean-Field-Game model.

The system is now :

$$\begin{cases} -\hbar_i \partial_t \Phi_i = \dfrac{\hbar_i^2}{2\mu_i}\Delta\Phi_i - g\left(\Phi_{3-i}\Gamma_{3-i}\right)\Phi_i \\ \hbar_i \partial_t \Gamma_i = \dfrac{\hbar_i^2}{2\mu_i}\Delta\Gamma_i - g\left(\Phi_{3-i}\Gamma_{3-i}\right)\Gamma_i \end{cases} \tag{31}$$

4.4 Nondimensionalization

To simplify the system of equations (30) and bring it into a dimensionless form, we can introduce dimensionless units.[7] In analogy with Bose-Einstein condensates, we can define the healing length as $v = \frac{\mu\sigma^4}{|g|}$, which represents the typical distance on which interactions balance quantum pressure (or diffusion in our case). Additionally, we define $\tau = \frac{2\mu^2\sigma^6}{g^2}$ as a typical time scale as in [5].

Let's focus on one dimensional case. By introducing dimensionless variables $t' = \frac{t}{\tau}$ and $x' = \frac{x}{v}$, we can rewrite (30) in a simplified form using these dimensionless coordinates :

$$\begin{cases} -\partial_{t'}\Phi = \partial_{x'x'}\Phi - 2vm\Phi \\ \partial_{t'}\Gamma = \partial_{x'x'}\Gamma - 2vm\Gamma \end{cases} \tag{32}$$

In this form, the healing length v appears as the only relevant parameter.

The above representation [Equations. (32) with all "primes" dropped] will be used for the rest of this paper.

5 Research work

The work presented in this section is an extension of a previous work conducted by Gabriel Rocheman in his report. [26]. In our study we introduce what we call a time discount in the value function, we will see how this will change the final results that Gabriel found in his internship.

5.1 The closer the clearer

The idea is to introduce a preference for a short time range, the player will see clearly in a close time range. To do so we need to add a discount factor under our cost functional integral. The new cost functional is now :

$$u(x,t) = \inf_{\overrightarrow{a}} \mathbb{E}\left[\int_t^T \left(\frac{\mu}{2}\left(\overrightarrow{a}_\tau\right)^2 - V[m]\left(\overrightarrow{X}_\tau\right)\right) e^{\gamma(t-\tau)} d\tau + c_T[m]\left(\overrightarrow{X}_T\right) e^{\gamma(t-T)}\right] \tag{33}$$

Where γ is a constant discount factor.

We will follow the same road we have followed in section 2.2.1 to obtain the Hamilton-Jacobi-Bellman Fokker- Planck's system (HJB-FP).

Denoting :

$$\mathcal{L}(\overrightarrow{x},\tau)[m] = \frac{\mu}{2}(\overrightarrow{a}(\tau)^2 - V[m](\overrightarrow{x},\tau))$$

Then we can write

$$u(\overrightarrow{x},t) = \inf_{\overrightarrow{a}} \left\{ \int_t^T \mathcal{L}(\overrightarrow{x},\tau)[m]e^{\gamma(t-\tau)}d\tau + e^{\gamma(t-T)}c_T\left(\overrightarrow{x}_T\right) \right\}$$
$$= \inf_{\overrightarrow{a}} \mathbb{E}\left\{ \int_t^{t+dt} \mathcal{L}(\overrightarrow{x},\tau)[m]e^{\gamma(t-\tau)}d\tau + e^{-\gamma dt}\left(\int_{t+dt}^T \mathcal{L}(\overrightarrow{x},\tau)[m]e^{\gamma(t+dt-\tau)}d\tau + e^{\gamma(t+dt-T)}c_T\left(\overrightarrow{x}_T\right)\right)\right\}$$

Then we use Bellman optimal principle [3] to write :

$$u(\overrightarrow{x},t) = \inf_{\overrightarrow{a}} \left[\mathcal{L}(\overrightarrow{x},t)dt + e^{-\gamma dt}u(\overrightarrow{x}+\overrightarrow{a}\,dt, t+dt)\right]$$
$$= \inf_{\overrightarrow{a}} \left[\mathcal{L}(\overrightarrow{x},t)dt + e^{-\gamma dt}\left(u(\overrightarrow{x},t) + \frac{\mathrm{d}}{\mathrm{d}t}u(\overrightarrow{x},t)dt\right)\right]$$

We can use Ito's chain rule to calculate the total derivative of u and by taking the limit dt $\to 0$, expanding the exponential term :

$$u(\overrightarrow{x},t) = \inf_{\overrightarrow{a}} \left\{\mathcal{L}(\overrightarrow{x},t)dt + (1-\gamma dt)\left[u(\overrightarrow{x},t) + dt\left(\partial_t u + \overrightarrow{a}\cdot\overrightarrow{\nabla}u + \frac{\sigma^2}{2}\Delta u\right)\right]\right\}$$

Then we keep order at first order in dt and simplifying we get :

$$0 = \partial_t u - V[m] + \frac{\sigma^2}{2}\Delta u - \gamma u + \inf_{\overrightarrow{a}}\left\{\frac{\mu}{2}\overrightarrow{a}^2 + \overrightarrow{a}\cdot\overrightarrow{\nabla}u\right\}$$

Which is what we will call the discounted HJB's equation.Finally by the same minimization done in section 2.1.3 and because Langevin's equation :

$$d\overrightarrow{X}_t = \overrightarrow{a}_t dt + \sigma d\overrightarrow{W}_t$$

did not change, hence the associated FP's equation neither, we can write the discounted HJB-FP's system :

$$\begin{cases} \partial_t u = -\frac{\sigma^2}{2}\Delta u + \frac{1}{2\mu}(\overrightarrow{\nabla}u)^2 + \gamma u + V[m] \\ u(\overrightarrow{x}, t=T) = c_T(\overrightarrow{x}) \\ \partial_t m = \frac{\sigma^2}{2}\Delta m + \frac{1}{\mu}\overrightarrow{\nabla}\cdot(m\overrightarrow{\nabla}u) \\ m(\overrightarrow{x}, t=0) = m_0(\overrightarrow{x}) \end{cases} \tag{34}$$

5.2 Discounted ergodic state

As we saw, we can have an ergodic state, and here there is still one but a little more general with the addition of the time discount. We thus can write :

$$u(\overrightarrow{x},t) \simeq u^e(\overrightarrow{x}) + f(t) \tag{35}$$

To determine f we will proceed by looking at the known $\gamma = 0$ case. [9]

For the non discounted problem, meaning $\gamma = 0$, such an ergodic state has been extensively studied, see [12], and it has been shown that $f_{\gamma=0}(t) = -\lambda t$, where λ is a constant.

The scope of this section is to study the ergodic state of an MFG for a generic γ.

Our hypothesis is that, in the case of $\gamma > 0$, the equation(35) defining the ergodic value function is still valid, thus, by plugging it in (19) we obtain :

$$\frac{\sigma^2}{2}\Delta u^e - \frac{1}{2\mu}\left(\overrightarrow{\nabla} u^e\right)^2 + f'(t) - \gamma\left(u^e(\overrightarrow{x}) + f(t)\right) = V[m] \tag{36}$$

then, in order to get rid of any explicit time dependence in the equation, and therefore in $u^e(\overrightarrow{x})$, we impose that $f'(t) = \gamma f(t) - \lambda$, which gives $f(t) = ke^{\gamma t} + \frac{\lambda}{\gamma}$, that we plug back in (35) to obtain that, for intermediate times, the ergodic state of the value function is :

$$u(\overrightarrow{x}, t) \simeq u^e(\overrightarrow{x}) + ke^{\gamma t} + \frac{\lambda}{\gamma}. \tag{37}$$

In order to fix k, we turn our attention to the behavior of the ergodic state for small values of the discount factor, i.e. when $\gamma \to 0$. In this case, we can write :

$$u(\overrightarrow{x}, t) \simeq u^e(\overrightarrow{x}) + k + k\gamma t + \frac{\lambda}{\gamma} \tag{38}$$

By taking $k = -\lambda/\gamma$ one recovers $f_{\gamma=0}(t) = -\lambda t$. The ergodic state of the value function for the discounted MFG is therefore :

$$u(x,t) \simeq u^e(x) + \frac{\lambda}{\gamma}\left(1 - e^{\gamma t}\right) = u^e(x) + f(t), \quad 0 << t << T \tag{39}$$

The equations followed by the position dependent part of the ergodic state both for the density and the value function for a general MFG model with a potential V not explicitly depending on time are then :

$$\begin{cases} \dfrac{\sigma^2}{2}\Delta u^e - \dfrac{1}{2\mu}\left(\overrightarrow{\nabla} u^e\right)^2 - \gamma u^e(x) - \lambda - V\left[m^e\right] = 0 \\ \dfrac{\sigma^2}{2}\Delta m^e + \dfrac{1}{\mu}\overrightarrow{\nabla} \cdot \left(m^e \overrightarrow{\nabla} u^e\right) = 0 \end{cases} \tag{40}$$

5.3 Cole Hopf transform

Let's return to our discounted system (34). Now we can apply a Cole Hopf transform and we land on the following system on (Γ, Φ) :

$$\begin{cases} \mu\sigma^2 \partial_t \Phi = -\frac{\mu\sigma^4}{2}\Delta\Phi - V[m]\Phi + \gamma\mu\sigma^2\Phi\log\Phi \\ \mu\sigma^2 \partial_t \Gamma = \frac{\mu\sigma^4}{2}\Delta\Gamma + V[m]\Gamma - \gamma\mu\sigma^2\Gamma\log\Phi \end{cases} \tag{41}$$

We place ourselves in the situation where V=-gm, we obtain :

$$\begin{cases} \mu\sigma^2\partial_t\Phi = -\dfrac{\mu\sigma^4}{2}\Delta\Phi + g\Gamma\Phi^2 + \gamma\mu\sigma^2\Phi\log\Phi \\ \mu\sigma^2\partial_t\Gamma = \dfrac{\mu\sigma^4}{2}\Delta\Gamma - g\Phi\Gamma^2 - \gamma\mu\sigma^2\Gamma\log\Phi \end{cases} \tag{42}$$

and with the same "nondimensionnalization" process explained in section 4.4 we get :

$$\begin{cases} \partial_t\Phi = -\Delta\Phi + 2vm\Phi + \gamma\Phi\log(\Phi) \\ \partial_t\Gamma = \Delta\Gamma - 2vm\Gamma - \gamma\Gamma\log(\Phi) \end{cases} \tag{43}$$

We can easily extend the previous equations (43) to a 2 populations system :

$$\begin{cases} \partial_t\Phi_i = -\Delta\Phi_i + 2v\left(\Phi_{3-i}\Gamma_{3-i}\right)\Phi_i + \gamma\Phi_i\log(\Phi_i) \\ \partial_t\Gamma_i = \Delta\Gamma_i - 2v\left(\Phi_{3-i}\Gamma_{3-i}\right)\Gamma_i - \gamma\Gamma_i\log(\Phi_i) \end{cases} \tag{44}$$

with i=1,2.

5.4 Homogeneous Solution

Now our goal is to study the stability of solutions of (44) with a perturbative treatment. We want to focus on an ergodic solution far from the edges of the game : $u(\overrightarrow{x},t) \simeq u^e(\overrightarrow{x}) + f(t)$ and $m = m^e$. Now proceeding to a Cole Hopf transformation on this :

$$\Phi = e^{-u/\mu\sigma^2} = e^{-u^e/\mu\sigma^2}e^{-f(t)/\mu\sigma^2} = \Phi^e e^{-f(t)/\mu\sigma^2}$$

and

$$\Gamma = \Gamma^e e^{f(t)/\mu\sigma^2}$$

with $\hbar = \mu\sigma^2$

Now as we will make a simplifying hypothesis, that u is homogeneous hence u^e does not depend on x. Then we can write $\Phi = Ae^{-f(t)/\hbar}$ and $\Gamma = Be^{f(t)/\hbar}$. Considering the condition $m = \Gamma\Phi$ then we must have $A = B = \sqrt{m}$.

In this order, we postulate a couple of homogeneous solutions :

$$\begin{cases} \Phi_i = \sqrt{m_i}e^{-f(t)/\hbar_i} \\ \Gamma_i = \sqrt{m_i}e^{f(t)/\hbar_i} \end{cases} \tag{45}$$

— With $\hbar_i = \mu_i \sigma_i^2$

Injecting these in our system (44) leads to the following dispersion relation :

$$2\nu m_{3-i} + \frac{\gamma}{2}\log(m_i) - \frac{\lambda}{\hbar_i} = 0 \tag{46}$$

Leading to the following expression for m_i :

$$m_i = e^{\frac{2\lambda}{\gamma\hbar_i}} e^{-4\nu m_{3-i}/\gamma}$$

5.4.1 Perturbative treatment

We now want to perturb the solution, denoting by :

$$\begin{cases} \Phi_p = (\sqrt{m} + \delta\Phi)e^{-f(t)/\hbar} = \Phi + \delta\Phi e^{-f(t)/\hbar} \\ \Gamma_p = (\sqrt{m} + \delta\Gamma)e^{f(t)/\hbar} = \Gamma + \delta\Gamma e^{f(t)/\hbar} \end{cases} \tag{47}$$

the perturbed solutions, with $\delta\Phi$ et $\delta\Gamma$ small perturbation. We will restrict ourselves to order 2 in $\delta\Phi$ and $\delta\Gamma$)

Let's inject these formulations into our system (44) :

— $\partial_t \Phi_i^p = \partial_t \Phi_i + \partial_t(\delta\Phi_i)e^{-f(t)/\hbar_i} - \frac{f'(t)}{\hbar_i}\delta\Phi_i e^{-f(t)/\hbar_i}$.

— $-\Delta\Phi_i^p = -\Delta\Phi_i + \overrightarrow{k}^2 \delta\Phi_i e^{-f(t)/\hbar_i}$.

— $2\nu\Gamma_{3-i}^p \Phi_{3-i}^p \Phi_i^p = 2\nu\Gamma_{3-i}\Phi_{3-i}\Phi_i + 2\nu\Gamma_{3-i}\Phi_{3-i}\delta\Phi_i e^{-f(t)/\hbar_i} + 2\nu\Gamma_{3-i}\Phi_i\delta\Phi_{3-i}e^{-f(t)/\hbar_{3-i}}$

$+ 2\nu\delta\Gamma_{3-i}\Phi_{3-i}\Phi_i e^{f(t)/\hbar_{3-i}} + O(\delta\Phi_i^2)$.

— $\gamma\Phi_i^p \log\Phi_i^p = \gamma\Phi_i\log(\Phi_i) + \gamma\delta\Phi_i e^{-f(t)/\hbar_i} + \gamma\delta\Phi_i e^{-f(t)/\hbar_i}\log(\Phi_i) + O(\delta\Phi_i^2)$.

We repeat the same method for Γ.

Then by using the fact that Φ and Γ are solution of system (44) and injecting the dispersion relation (46), we obtain the following matrix equation : $\partial_t \overrightarrow{\delta} = \mathbb{A} \cdot \overrightarrow{\delta}$ with $\overrightarrow{\delta} = (\delta\Phi_1, \delta\Gamma_1, \delta\Phi_2, \delta\Gamma_2)$

And

$$\mathbb{A} = \begin{pmatrix} \overrightarrow{k}^2 + \gamma & 0 & 2\nu\sqrt{m_1 m_2} & 2\nu\sqrt{m_1 m_2} \\ -\gamma & -\overrightarrow{k}^2 & -2\nu\sqrt{m_1 m_2} & -2\nu\sqrt{m_1 m_2} \\ 2\nu\sqrt{m_1 m_2} & 2\nu\sqrt{m_1 m_2} & \overrightarrow{k}^2 + \gamma & 0 \\ -2\nu\sqrt{m_1 m_2} & -2\nu\sqrt{m_1 m_2} & -\gamma & -\overrightarrow{k}^2 \end{pmatrix} \tag{48}$$

Let denote $\theta = 2\nu\sqrt{m_1 m_2}$ then :

$$\mathbb{A} = \begin{pmatrix} \overrightarrow{k}^2 + \gamma & 0 & \theta & \theta \\ -\gamma & -\overrightarrow{k}^2 & -\theta & -\theta \\ \theta & \theta & \overrightarrow{k}^2 + \gamma & 0 \\ -\theta & -\theta & -\gamma & -\overrightarrow{k}^2 \end{pmatrix} \tag{49}$$

Now we want to know if the perturbed solutions are stable, this is equivalent to $\overrightarrow{\delta} \to 0$ when $t \to \inf$. With Matematica, we can find the eigenvalues of $\mathbb{A}$.

We can see that we are recovering the same matrix as in [26] in the limit $\gamma \to 0$ which is reassuring.

Here they are :

$$\begin{cases} \lambda_1 = \frac{\gamma}{2}\left(1 - \sqrt{1 + \frac{4\overrightarrow{k}^2}{\gamma}(1 + \frac{\overrightarrow{k}^2}{\gamma}) - \frac{8\overrightarrow{k}^2\theta}{\gamma^2}}\right) \\ \lambda_2 = \frac{\gamma}{2}\left(1 - \sqrt{1 + \frac{4\overrightarrow{k}^2}{\gamma}(1 + \frac{\overrightarrow{k}^2}{\gamma}) + \frac{8\overrightarrow{k}^2\theta}{\gamma^2}}\right) \\ \lambda_3 = \frac{\gamma}{2}\left(1 + \sqrt{1 + \frac{4\overrightarrow{k}^2}{\gamma}(1 + \frac{\overrightarrow{k}^2}{\gamma}) - \frac{8\overrightarrow{k}^2\theta}{\gamma^2}}\right) \\ \lambda_4 = \frac{\gamma}{2}\left(1 + \sqrt{1 + \frac{4\overrightarrow{k}^2}{\gamma}(1 + \frac{\overrightarrow{k}^2}{\gamma}) + \frac{8\overrightarrow{k}^2\theta}{\gamma^2}}\right) \end{cases}$$

We can plot these 4 eigenvalues to better visualize their behavior for different values of γ and θ.

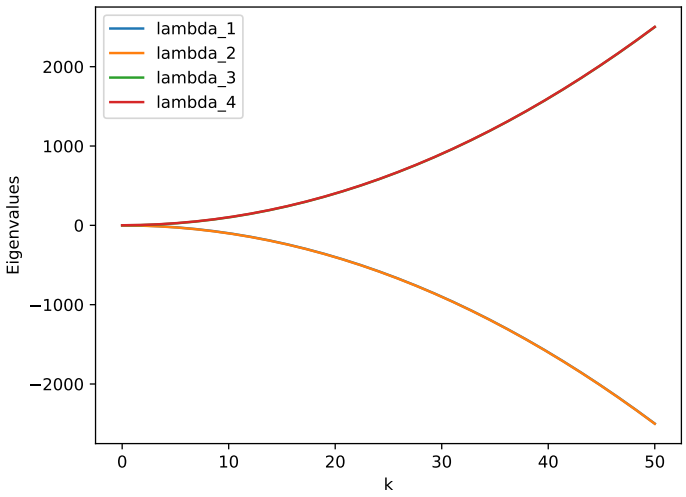

FIGURE 1 – Plot of eigenvalues of $\mathbb{A}$ for $\gamma = \theta = 1$

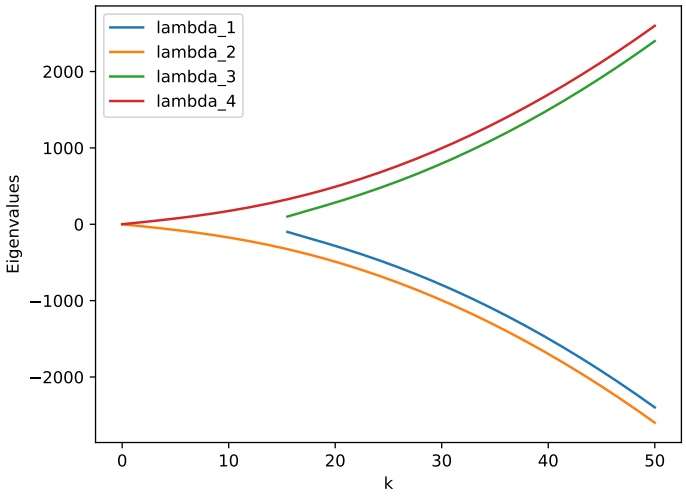

FIGURE 2 – Plot of eigenvalues of $\mathbb{A}$ for $\gamma = 1$ and $\theta = 100$

5.4.2 Physicist intuition

The eigenvalues shown in Figure 1 may seem surprising at first glance. In classical physical models, a large positive eigenvalue indicates strong mode growth. However, in this case, there is no value of $\overrightarrow{k}$ for which the eigenvalues are singular or maximal, which would suggest a preferred $\overrightarrow{k}$. On the contrary, interpreting the graph according to the physicist's usual intuition, we would expect to see four modes : two decreasing modes and two exponentially growing modes, whose growth seems to increase as $\overrightarrow{k}$ becomes large (i.e. for smaller wave-

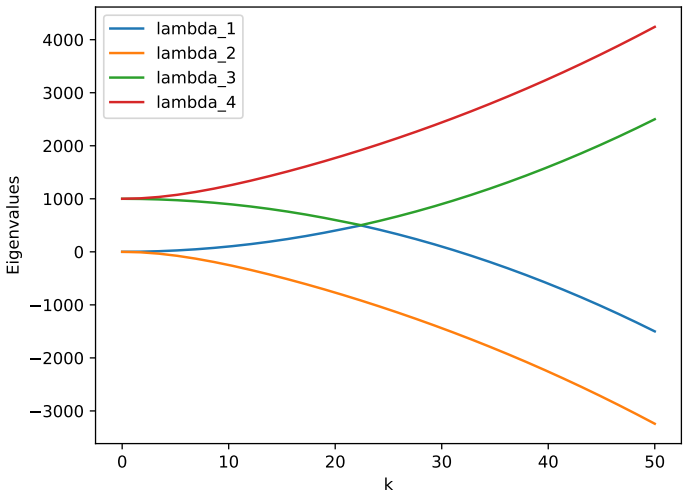

FIGURE 3 – Plot of eigenvalues of $\mathbb{A}$ for $\gamma = 1000$ and $\theta = 1000$

lengths).

To understand this phenomenon further, it is necessary to take the analysis a step further. It seems unlikely that the system favors disturbances with infinitely small wavelengths.

5.5 Decoupling of initial and final perturbations

Let's take a qualitative line of reasoning to recover our intuition of the problem.

Intuitively, we can imagine that the smaller the wavelength of the perturbation (i.e. the larger $\overrightarrow{k}$), the greater the influence of noise, as in a scattering problem. So, beyond a certain value of $\overrightarrow{k}$, noise will attenuate our perturbation, and the homogeneous solution will be stable. However, the final condition introduces a certain complexity : how does the stability of the homogeneous solution manifest itself in this case ?

Given the form of the equation, we expect to observe exponential attenuation when the solution is stable, whether for the final or initial conditions. Consequently, the final perturbation, $\delta\Phi_{i|t=T}$, should exhibit exponential attenuation backward (i.e. exponential growth from the past to the future). We can therefore imagine that if T is not too small, $\delta\Phi_{i|t=0}$ will be close to zero. Similarly, the initial condition, $\delta\Gamma_{i|t=0} + \delta\Phi_{i|t=0} \simeq \delta\Gamma_{i|t=0}$ will be attenuated toward the future.

The problem is then greatly simplified : we have an initial condition on the $\delta\Gamma_i$ attenuated towards the future and a final condition on the $\delta\Phi_i$, attenuated towards the past, so the $\delta\Phi_i$

are negligible when the $\delta\Gamma_i$ are not and vice versa. We should therefore observe a decoupling of the problem on degrees of freedom propagated toward the future and those propagated toward the past.

5.6 Stability analysis

5.6.1 Eigenvalues of the decoupled system

Algebraically, this decoupling translates into the separation of our vector space into two subspaces $E_\Phi = \text{Vect}\{(1,0,0,0),(0,0,1,0)\}$ and $E_\Gamma = \text{Vect}\{(0,1,0,0),(0,0,0,1)\}$: we neglect the $\mathbb{A}$ matrix elements that couple the $\delta\Phi_i$ to the $\delta\Gamma_j$.

After making the corresponding simplifications in the linear system, we obtain two subsystems of two equations, each with a linear coupling : one for the $\delta\Phi_i$ and one for the $\delta\Gamma_i$, with the respective matrices $\mathbb{A}_\Phi$ and $\mathbb{A}_\Gamma$:

$$\mathbb{A}_\Phi = \begin{pmatrix} \overrightarrow{k}^2 + \gamma & \theta \\ \theta & \overrightarrow{k}^2 + \gamma \end{pmatrix} \tag{50}$$

$$\mathbb{A}_\Gamma = \begin{pmatrix} -\overrightarrow{k}^2 & -\theta \\ -\theta & -\overrightarrow{k}^2 \end{pmatrix} \tag{51}$$

Let's write explicit equations :

$$\begin{cases} \partial_t \overrightarrow{\delta\Phi} = \mathbb{A}_\Phi \overrightarrow{\delta\Phi} \\ \partial_t \overrightarrow{\delta\Gamma} = \mathbb{A}_\Gamma \overrightarrow{\delta\Gamma} \end{cases} \tag{52}$$

Diagonalizing A_Φ means projecting onto $(1,1)$ and $(1,-1)$, i.e. taking the sum and difference of each perturbation. In other words, it means separating the part of the perturbation that is in phase ($\delta\Phi_1 = \delta\Phi_2$) from the part that is in phase opposition ($\delta\Phi_1 = -\delta\Phi_2$).

Eigenvalues of A_Φ and A_Γ are :

$$\begin{cases} v_{\Phi_\pm} = \overrightarrow{k}^2 + \gamma \pm \theta \\ v_{\Gamma_\pm} = -\overrightarrow{k}^2 \pm \theta \end{cases} \tag{53}$$

Looking at (52), to have $\delta\Phi$ attenuated towards the past, v_Φ must be positive and $\delta\Gamma$ attenuated towards the future, v_Γ must be negative.

We now want to check that the eigenvalues are indeed verifying those conditions :

1. Perturbation in phase :
   Associated eigenvalues : $v_{\Phi_+} = \overrightarrow{k}^2 + \gamma + \theta$ and $v_{\Gamma_-} = -\overrightarrow{k}^2 - \theta$
   It is trivial that v_{Φ_+} is positive and v_{Γ_-} is negative, so this is consistent.

2. Perturbation in phase opposition :
   Associated eigenvalues : $v_{\Phi_-} = \overrightarrow{k}^2 + \gamma - \theta$ and $v_{\Gamma_+} = -\overrightarrow{k}^2 + \theta$
   This is more subtle we need to examine different sub cases :
   — If $v_{\Phi_-} > 0$ then $\overrightarrow{k}^2 > \theta - \gamma$
   hence $v_{\Gamma_+} = \theta - \overrightarrow{k}^2 < \gamma$.
   Unfortunately we have $v_{\Gamma_+} < 0$ only if $\gamma = 0$.

   But the reciprocal proposition is not true :
   If $v_{\Gamma_+} < 0$ then $\overrightarrow{k}^2 > \theta > \theta - \gamma$ so $v_{\Phi_-} > 0$,
   and we are consistent again.
   — If $v_{\Phi_-} < 0$ then $\overrightarrow{k}^2 < \theta - \gamma < \theta$,
   hence $v_{\Gamma_+} > 0$.
   Perturbation now grows and we are not consistent again.

We expect to be able to distinguish two regimes : above a certain value of $\overrightarrow{k}^2$, initial and final disturbances decouple and attenuate, whereas below, disturbances may increase.

We're now going to calculate precisely the different evolution regimes of a perturbation of a certain wave number, which we identify with the parameter $\overrightarrow{k}^2$.

5.6.2 Eigenvalues of $\mathbb{A}$

We recall the eigenvalues of $\mathbb{A}$:

$$\begin{cases} \lambda_1 = \frac{\gamma}{2}\left(1 - \sqrt{1 + \frac{4\overrightarrow{k}^2}{\gamma}(1 + \frac{\overrightarrow{k}^2}{\gamma}) - \frac{8\overrightarrow{k}^2\theta}{\gamma^2}}\right) \\ \lambda_2 = \frac{\gamma}{2}\left(1 - \sqrt{1 + \frac{4\overrightarrow{k}^2}{\gamma}(1 + \frac{\overrightarrow{k}^2}{\gamma}) + \frac{8\overrightarrow{k}^2\theta}{\gamma^2}}\right) \\ \lambda_3 = \frac{\gamma}{2}\left(1 + \sqrt{1 + \frac{4\overrightarrow{k}^2}{\gamma}(1 + \frac{\overrightarrow{k}^2}{\gamma}) - \frac{8\overrightarrow{k}^2\theta}{\gamma^2}}\right) \\ \lambda_4 = \frac{\gamma}{2}\left(1 + \sqrt{1 + \frac{4\overrightarrow{k}^2}{\gamma}(1 + \frac{\overrightarrow{k}^2}{\gamma}) + \frac{8\overrightarrow{k}^2\theta}{\gamma^2}}\right) \end{cases}$$

λ_2 and λ_4 are always well defined because everything is positive under the square root hence there are no difficulties. However it's not trivial for λ_1 and λ_3

Now we have different sub-cases to study the sign of the "square rooted term" in λ_1 and λ_3. We need this condition :

$$1 + \frac{4\overrightarrow{k}^2}{\gamma}(1 + \frac{\overrightarrow{k}^2}{\gamma}) - \frac{8\overrightarrow{k}^2\theta}{\gamma^2} > 0 \tag{54}$$

And for that, we introduce the following function to study :

$$h(x) = x^4 + (\gamma - 2\theta)x^2 + \frac{\gamma^2}{4} \tag{55}$$

This function can be expressed in a more suitable way by introducing the new variable : $X = x^2 = \overrightarrow{k}^2$.

We can now express h with this new variable and study : $X^2 + (\gamma - 2\theta)X + \frac{\gamma^2}{4}$.

Let's solve this equation in order to find when this quantity is indeed positive :

Let's denote the discriminant $\Delta = 4\theta(\theta - \gamma)$.

By looking at it there are immediately 3 sub cases emerging :

— $\Delta < 0$ ie $\gamma > \theta$: there are no real solutions,and the function is always positive, so eigenvalues are defined and we are satisfied.

— $\Delta = 0$ ie $\gamma = \theta$: The equation admit one real solution denoted $X_0 = \gamma/2$ ie $\overrightarrow{k}^2 = \gamma/2$. We must retain only the positive solution denoted $\overrightarrow{k_0} = \sqrt{\frac{\gamma}{2}}$. In this case, the function is always positive or null, so eigenvalues are defined and everything is indeed fine.

— $\Delta > 0$ ie $\gamma < \theta$: In this case there is 2 real solutions $X_\pm = \theta - \frac{\gamma}{2} \pm \sqrt{\theta(\theta - \gamma)}$ and we might have negative values for h.

Let's focus on the last case $\Delta > 0$:

We retain only positive solutions. Numerically we can check that X_- is always positive for any value of θ and γ. We can then take the square root of X_- and the valid solutions are then :

— $\overrightarrow{k_+} = \sqrt{X_+}$

— $\overrightarrow{k_-} = \sqrt{X_-}$

Let's sum up what we learned from the study of eigenvalues :

— If $\gamma \geq \theta$ All eigenvalues are defined, the solutions are stable, and perturbations are decaying.

— If $\gamma < \theta$: λ_1 and λ_2 are defined if :

  — $\overrightarrow{k} < \overrightarrow{k_-}$

  — $\overrightarrow{k} > \overrightarrow{k_+}$

5.7 Physical interpretation

In the situation where $\theta \ll \gamma$, implies that the agent has limited foresight and is not strongly interacting with others, the homogeneous solution is stable. This means that small perturbations around the equilibrium will eventually dampen out, and the system will converge to a stable state.

On the other hand, in the situation where $\gamma \ll \theta$, which implies that the agent has a strong ability to look far ahead in the future and is strongly interacting with others, there can be problems and difficulties in defining a solution. This is because in this case, the system may exhibit uncontrolled growing perturbations, leading to mathematical challenges in characterizing the dynamics. From a physical perspective, this suggests that the agents' strategies and interactions are such that the system becomes more volatile and sensitive to perturbations, potentially leading to instability or oscillatory behavior. We could imagine that in this case, the agent can see the terminal cost and will be torn between avoiding strangers and searching for the minimizing position for terminal cost. This may be explaining the instability.

6 Conclusion

In conclusion, the choice of parameters γ and θ in the MFG framework can have significant implications for the stability and behavior of the system. We have shown the existence of two different regimes and this analysis provides insights into the dynamics of the mean field game and highlights the importance of considering the interplay between agents' foresight and interactions.

Moving forward, it would be interesting to extend this work and study a cost function closer to the original Schelling model. By incorporating different cost functions, it is possible to explore and analyze various scenarios and their impact on the behavior and stability of the system. This could further enhance our understanding of mean field games and their applications in social dynamics.

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